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# Majorization and its Applications on Some Functions

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## ABSTRACT

This paper reviews the special case of an order which is called Majorization ordering. It generalizes vector Majorization and some applications that have come after the publication of Marshall and Olkin Inequalities. It presents basic properties of Majorization and two important kinds of Majorization which are Weakly Supermajorization and Weakly Submajorization and some relations between them. Furthermore, this paper also contains maps from $\mathbb{R}^n$ to $\mathbb{R}^m$ which preserve various orders that most of these orders are elementary and useful characterizations of Majorization, as Majorization together with the strongly related concept of schur-convexity gives an important characterization of convex functions that expresses preservation of order rather than convexity. Also in this study, examples are used to explore the characteristics of majorization, weakly supermajorization, and weakly submajorization as well as the relationships between them. We described the application of majorization on various functions, such as monotonic functions, convex functions, and so on, with some properties by taking into account the concept of our title majorization and its applications on some Functions. Theorems and examples are used to explain such outcomes.

**Keywords:** Monotonic function, Isotone, Substochastic matrix, Weakly supermajorization, and Submajorization.

## INTRODUCTION:

Comparison of two vectors sometimes leads to interesting inequalities, which is an important tool in comparing vectors. In general it is said that vectors are not comparable. They are either equal or unequal, yes we have been comparing the vectors in terms of their norms, but the readers will learn a new type of comparison of vectors in term their co-ordinates, and how it will be applied on functions. Majorization theory provides a method to compare the vectors with same number of co-ordinates. I hope that the result in this work will lead readers to discover further applications and extensions.

For this Rajendra Bhatia has given details in the “Matrix Analysis" (Bhatia, 1997). If we talk about history of majorization, indeed many of the key ideas related to majorization were discussed in the volume entitled “Inequalities” by Hardy, Littlewood and Polya (Hardy. E. & G. Polya. 1934). The appearance of Marshall and Olkin (W. Marshal. & Olkin. 1979) book on inequalities with special emphasize on majorization generated a surge of interest in majorazation and Schur Convexity. probability of covering the circle by random arcs, by Shepp and Huffer (Huffer & A Shepp, 1987) effect of unequal catchability on estimates of the number of classes in a population by Nayak and Christman (Nayak & Christman, 1992) the mean waiting time for a pattern probability by Ross (Ross, 1999). Balinski and Young provided a good survey of the methods usually considered in (L. Balinski and young, 2010).

Martin Mittelbach and Eduard Jorswieck applied majorization theory to compare different tap correlation scenarios for perfect, partial, and no CSI at the transmitter (Mittelbech, 2008) Eduard Jorswieck and Martin Mittelbach used majorization for functions to show that the average rate with perfectly informed receiver is largest for uncorrelated scattering if the transmitter is uninformed (lorseck & Mittelbech, 2009).

**Basic Notations and Preliminaries**

Majorization is an important order notion that arises in several areas of mathematics. The following notations will be used in the subsequent discussion.

Let $a \in \mathbb{R}^n$, then $a^{\uparrow}$ and $a^{\downarrow}$ denotes the vectors which obtained by rearranging the co-ordinates of $a$ in increasing order and decreasing order respectively,

Where,

$\mathbb{R}^n$ Shows set of $n$ tuples.

For any two real numbers $a$ and $b$, maximum of $a$ and $b$, and minimum of $a$ and $b$ are denoted as $(a \vee b)$ and $(a \wedge b)$ respectively.

**Definition 2.1**

For any real number $a \in \mathbb{R}$ the function $a^+$, replaces the negative real number to 0, that is $a^+ = a \vee 0$.

**Definition 2.2**

For any real number $a \in \mathbb{R}$, $|a| = a \vee (-a)$.

Now we will extend these to

$\mathbb{R}^n$.

Let $a, b \in \mathbb{R}^n$,$a \wedge b = (a_1 \wedge b_1, a_2 \wedge b_2, \dots, a_n \wedge b_n)$.

Now for $a \in \mathbb{R}^n$, the function $a^+$ replaces the negative co-ordinates of $a$ by $0$. that is $a^+ = (a_1 \vee 0, a_2 \vee 0, \dots, a_n \vee 0)$.

Also for any $a \in \mathbb{R}^n$,

$|a| = (a_1 \vee (-a_1), a_2 \vee (-a_2), \dots, a_n \vee (-a_n))$.

**Definition 2.3**

**Majorization**

Let $a, b \in \mathbb{R}^n$, we call $a$ is majorized by $b$ in symbol $a \prec b$, if

$$a \prec b \Rightarrow \sum_{i=1}^{n} a_i = \sum_{i=1}^{n} b_i \ and \ \sum_{i=1}^{k} a_i^{\downarrow} \leq \sum_{i=1}^{k} b_i^{\downarrow} \ for \ 1 \leq k \leq n$$

In case $a, b$ are in decreasing order.

The order of the entries of the vectors does not affect the majorization that is if we arrange the vectors in increasing order, then

$$\sum_{i=1}^{n} a_i^{\uparrow} = \sum_{i=1}^{n} b_i^{\uparrow} \ and \ \sum_{i=1}^{k} a_i^{\uparrow} \geq \sum_{i=1}^{k} b_i^{\uparrow} \ \forall \ 1 \leq k \leq n$$

Let $e$ denotes the vector $(1,1,1,\dots,1)$ and for any subset $I$ of $\{1,2,3,\dots,n\}$, let $e_j$ denotes the vector whose $j^{th} =$ component is 1 if $j \in I$ and 0 if $j \notin I$. Given a vector $a \in \mathbb{R}^n$,

Then $$tr\ a = \sum_{i=1}^{n} a_i = \langle a, e \rangle,$$

Where $\langle \cdot, \cdot \rangle$ denotes the inner product in $\mathbb{R}^n$ and $tr$ stands for trace.

$$\sum_{i=1}^{k} a_i^{\downarrow} = \max_{|I|=k} \langle a, e_I \rangle,$$

Also $a \prec b$ if and only for each subset $I$ of $\{1,2,3,\ldots,n\}$, there exist a subset $J$ with $|I| = |J|$, such that $\langle a, e_I \rangle = \langle b, e_J \rangle$ and tr a=tr b

**Definition 2.4**

Let $a, b \in \mathbb{R}^n$, we call $a$ is weakly submajorized by $b$ in symbol $a \prec_w b$, if

$$\sum_{=1}^{k} a_i^{\downarrow} \leq \sum_{i=1}^{k} b_i^{\downarrow} \text{ for } 1 \leq k \leq n.$$

**Definition 2.5**

Let $a, b \in \mathbb{R}^n$, we call $a$ is weakly supermajorized by $b$ in symbol $a \prec^w b$, if

$$\sum_{i=1}^{k} a_i^{\uparrow} \geq \sum_{i=1}^{k} b_i^{\uparrow} \text{ for } 1 \leq k \leq n.$$

**Definition 2.6**

A real valued function $f$ from $\mathbb{R}^n$ to $\mathbb{R}^m$ be isotone if

$$a \prec b \Rightarrow f(a) \prec_w f(b)$$

**Definition 2.7**

A real valued function $f$ from $\mathbb{R}^n$ in to $\mathbb{R}^m$ be strongly isotone if

$$a \prec_w b \Rightarrow f(a) \prec_w f(b)$$

**Definition 2.8**

Function $f$ from $\mathbb{R}^n$ in to $\mathbb{R}^m$ be strictly isotone if

$$a \prec \quad b \Rightarrow f(a) \prec \quad f(b)$$

More ever if $m = 1$, we have $f: \mathbb{R}^n \to \mathbb{R}$, then isotone maps are precisely schur convex maps.

**Lemma 2.9**

For $a, b \in \mathbb{R}^n$ the following statements are equivalent.

1. $a \prec b$.
2. $a$ is obtained from $b$ by a finite number of $T$ −transformations.
3. $a$ is in the convex hull of all vectors obtained by permuting the co-ordinates of $b$.
4. $a = Ab$ for some doubly stochastic matrix $A$.

**Lemma 2.10**

Let $a, b$ be two vectors with non-negative co-ordinates, then $a \prec_w b$ if and only if $a = Qb$ for some doubly substochastic matrix $Q$

## RESULTS:

In some other terms the following import-ant results about the properties of majorization, weakly submajorization and weakly supermajorization and relations between them was established in (Bhatia, 1997). We provide the proof for convenience.

This theorem will show the relation of majorization, weakly submajorization and weakly supermajorization

**Theorem 3.1**

For $a, b \in \mathbb{R}^n$

1. $a \prec b \Leftrightarrow a \prec_w b, a \prec^w b$.
2. $a \prec_w b \Rightarrow \alpha a \prec_w \alpha b$ and $a \prec^w b \Rightarrow \alpha a \prec^w \alpha b$, for every $\alpha > 0$.
3. $a \prec_w b \Leftrightarrow -a \prec^w - b$.

4. $a \prec b \Rightarrow \alpha a \prec \alpha b$, For every $\alpha \in \mathbb{R}$.

**Proof 1**

Let $a \prec b$, then by definition we have

$$\sum_{i=1}^{n} a_i = \sum_{i=1}^{n} b_i \ and \ \sum_{i=1}^{k} a_i^{\downarrow} \leq \sum_{i=1}^{k} b_i^{\downarrow} \ for \ 1 \leq k \leq n$$

$$\Rightarrow a \prec_w b.$$

Also if $a \prec b$, then by definition we have

$$\sum_{i=1}^{n} a_i^{\uparrow} = \sum_{i=1}^{n} b_i^{\uparrow} \ and \ \sum_{i=1}^{k} a_i^{\uparrow} \geq \sum_{i=1}^{k} b_i^{\uparrow} \ \forall \ 1 \leq k \leq \Rightarrow a \overset{w}{\prec} b.$$

$$\Rightarrow a \prec^{w} b$$

Conversely,

Let $a \prec_w b$ and $a \prec^w b$.

To prove $a \prec b$, when $a \prec^w b$, this implies

$$\sum_{i=1}^{k} a_i^{\uparrow} \geq \sum_{i=1}^{k} a_i^{\uparrow} \tag{1}$$

And we have the relation

$$\sum_{i=1}^{k} a_i^{\uparrow} = \sum_{i=1}^{n} a_i - \sum_{i=1}^{n-k} a_i^{\downarrow} \tag{2}$$

Similarly for $b$ we have

$$\sum_{i=1}^{k} b_i^{\uparrow} = \sum_{i=1}^{n} b_i - \sum_{i=1}^{n-k} b_i^{\downarrow} \tag{3}$$

Using (2) and (3) in equation (1) we have

$$\sum_{i=1}^{n} a_i - \sum_{i=1}^{n-k} a_i^{\downarrow} \geq \sum_{i=1}^{n} b_i - \sum_{i=1}^{n-k} b_i^{\downarrow},$$

Since we have $a \prec_w b$, that is

$$\sum_{i=1}^{n-k} a_i^{\uparrow} \geq \sum_{i=1}^{n-k} a_i^{\uparrow} \Rightarrow \sum_{i=1}^{n} a_i \geq \sum_{i=1}^{n} b_i. \tag{4}$$

Similarly by $a \prec_w b$ we can show that

$$\sum_{i=1}^{n} a_i \leq \sum_{i=1}^{n} b_i. \tag{5}$$

From equations (4) and (5) we have

$$\sum_{i=1}^{n} a_i = \sum_{i=1}^{n} b_i,$$

This result together with the given conditions imply $a \prec b$.

**(2)** Let $a \prec^w b$, to show
$\alpha a \prec^w \alpha b$.
$a \prec^w b$, implies

$$\sum_{i=1}^{k} a_i^{\uparrow} \geq \sum_{i=1}^{k} b_i^{\uparrow}.$$

Now for $\alpha > 0$ we have

$$\alpha \sum_{i=1}^{k} a_i^{\uparrow} \geq \alpha \sum_{i=1}^{k} b_i^{\uparrow} \Rightarrow \sum_{i=1}^{k} \alpha a_i^{\uparrow} \geq \sum_{i=1}^{k} \alpha b_i^{\uparrow},$$

Also let $a \prec_w b$, which implies

$$\sum_{i=1}^{k} a_i^{\downarrow} \leq \sum_{i=1}^{k} b_i^{\downarrow},$$

Now for $\alpha > 0$ we have

$$\alpha \sum_{i=1}^{k} a_i^{\downarrow} \leq \alpha \sum_{i=1}^{k} b_i^{\downarrow} \Rightarrow \sum_{i=1}^{k} \alpha a_i^{\downarrow} \leq \sum_{i=1}^{k} \alpha b_i^{\downarrow} \Rightarrow \alpha a \prec_w \alpha b.$$

Note that $(-a)_i^{\uparrow} = -a_i^{\downarrow}$ using this, we see that $-a \prec^w - b$,

$$\sum_{i=1}^{k} (-a_i)^{\uparrow} \geq \sum_{i=1}^{k} (-\mathrm{b}_i)^{\uparrow}$$

$$\sum_{i=1}^{n} a_i = \sum_{i=1}^{n} a_i \; and \; \sum_{i=1}^{k} a_i^{\downarrow} \leq \sum_{i=1}^{k} a_i^{\downarrow} \quad \text{for } 1 \leq k \leq n \Rightarrow a \prec a.$$

$$\Leftrightarrow \sum_{i=1}^{k} -a_i^{\downarrow} \geq \sum_{i=1}^{k} -b_i^{\downarrow}$$

$$\Leftrightarrow \sum_{i=1}^{k} a_i^{\downarrow} \leq \sum_{i=1}^{k} b_i^{\downarrow} \Leftrightarrow a \prec_w b$$

(4) If $a \prec b$, which implies

$$\sum_{i=1}^{k} a_i^{\downarrow} \leq \sum_{i=1}^{k} b_i^{\downarrow} \text{ for } 1 \leq k \leq n \text{ and } \sum_{i=1}^{n} a_i = \sum_{i=1}^{n} b_i.$$

To prove $\alpha a \prec \alpha b$, we multiply both sides of the last two equations by $\alpha \geq 0$,

$$\alpha \sum_{i=1}^{k} a_i^{\downarrow} \leq \alpha \sum_{i=1}^{k} b_i^{\downarrow} \text{ and } \alpha \sum_{i=1}^{n} a_i = \alpha \sum_{i=1}^{n} b_i$$

$$\sum_{i=1}^{k} \alpha\, a_i^{\downarrow} \leq \sum_{i=1}^{k} \alpha\, b_i^{\downarrow}, \text{ for } 1 \leq k \leq n$$

$$\sum_{i=1}^{n} \alpha\, a_i = \sum_{i=1}^{n} \alpha\, b_i \Rightarrow \alpha a \prec \alpha b.$$

Now again consider $a \prec b$, which implies

$$\sum_{i=1}^{k} a_i^{\downarrow} \leq \sum_{i=1}^{k} b_i^{\downarrow} \text{ for } 1 \leq k \leq n \text{ and } \sum_{i=1}^{n} a_i = \sum_{i=1}^{n} b_i\,.$$

Multiplying both sides by $\alpha < 0$,
We get,

$$\alpha \sum_{i=1}^{k} a_i^{\downarrow} \geq \alpha \sum_{i=1}^{k} b_i^{\downarrow} \text{ for } 1 \leq k \leq n \Rightarrow \sum_{i=1}^{k} \alpha\, a_i^{\uparrow} \geq \sum_{i=1}^{k} \alpha\, b_i^{\uparrow}$$

$$\text{and} \quad \alpha \sum_{i=1}^{n} a_i = \alpha \sum_{i=1}^{n} b_i \Rightarrow \sum_{i=1}^{n} \alpha\, a_i = \sum_{i=1}^{n} \alpha\, b_i \Rightarrow \alpha a \prec \alpha b.$$

**Theorem 3.2**
The relation of majorization, weakly supermajorization and weakly submajorization are reflexive and transitive.
**Proof 1**
Let $a \in \mathbb{R}^n$, if we rearrange $a$ in decreasing order, then we have

$\sum_{i=1}^{n} \alpha\, a_i = \sum_{i=1}^{n} \alpha\, b_i$ and $\sum_{i=1}^{k} a_i^{\downarrow} \leq \sum_{i=1}^{k} b_i^{\downarrow}$ for $1 \leq k \leq n \Rightarrow a \prec \alpha$

So $\prec$ is reflexive, from this we can conclude that $\prec_w$ and $\prec^w$ are also reflexive.
Now let $a^{\downarrow} = (a_1, a_2, \ldots, a_n)$, $b^{\downarrow} = (b_1, b_2, \ldots, b_n)$ and $c^{\downarrow} = (c_1, c_2, \ldots, c_n)$ in decreasing order with $a \prec b$ and $b \prec c$, we want to show that
$a \prec c$.
Since $a \prec b$, this implies

$$\sum_{i=1}^{n} a_i = \sum_{i=1}^{n} b_i \text{ and } \sum_{i=1}^{k} a_i^{\downarrow} \leq \sum_{i=1}^{k} b_i^{\downarrow} \text{ for } 1 \leq k \leq n. \quad (6)$$

Also, since $b \prec c$ this implies

$$\sum_{i=1}^{n} b_i = \sum_{i=1}^{n} c_i \text{ and } \sum_{i=1}^{k} b_i^{\downarrow} \leq \sum_{i=1}^{k} c_i^{\downarrow} \text{ for } 1 \leq k \leq n. \quad (7)$$

From equations (6) and (7) we have

$$\sum_{i=1}^{n} a_i = \sum_{i=1}^{n} c_i \text{ and } \sum_{i=1}^{k} a_i^{\downarrow} \leq \sum_{i=1}^{k} c_i^{\downarrow} \text{ for } 1 \leq k \leq n\,.$$

This shows $a \prec c$, which implies $\prec$ is transitive. Hence $\prec^w$ and $\prec_w$ are also transitive.
As we have already seen that the relations of majorization, weakly submajorization and weakly supermajorization are all reflexive and transitive. Next, we see that none of these are a partial order relation.
For e.g.:
Let
$a = (1,2,3,4,5)$ and
$b = (3,2,1,5,4)$,
Clearly

$a \neq b$.
One can easily check that
$a \prec b$ And

$b \prec a$.

Same example will work for weakly submajorization and weakly supermajorization also. In fact, for two vectors $a$ and $b$, if $a$ is obtained by permuting the co-ordinates of $b$, one can easily see that $a \prec b$ and vice versa but they are not equal vectors, also none of these are symmetric, clearly $\prec^w$ and $\prec_w$ are not symmetric too, the symmetric relation is only possible, if $a = pb$ for some permutation matrix $p$, otherwise $\prec$ is anti-symmetric, reflexive and transitive on the set $\{a \in \mathbb{R}^n : a_1 \geq a_2 \geq \ldots \geq a_n\}$.

**Theorem 3.3**
**Some Equivalent Properties**

Let $a, b \in \mathbb{R}^n$, then

1. $a \prec_w b$ if and only if for all $t \in \mathbb{R}$

$$\sum_{i=1}^{n} (a_i - t)^+ \leq \sum_{i=1}^{n} (b_i - t)^+.$$

2. $a \prec^w b$ if and only if for all $t \in \mathbb{R}$

$$\sum_{i=1}^{n} (t - a_i)^+ \leq \sum_{i=1}^{n} (t - b_i)^+.$$

3. $a \prec b$ if and only if for all $t \in \mathbb{R}$

$$\sum_{i=1}^{n} |a_i - t| \leq \sum_{i=1}^{n} |b_i - t|.$$

**Proof 1**
Let $a \prec_w b$, that is

$$\sum_{i=1}^{k} a_i^{\downarrow} \leq \sum_{i=1}^{k} b_i^{\downarrow}.$$

Now if $t > a_i^{\downarrow}$ for each
$i$,
Then
$(a_i - t)^+ = 0$,
For each $i$.

Hence

$$\sum_{i=1}^{n} (a_i - t)^+ \leq \sum_{i=1}^{n} (b_i - t)^+.$$

Now let $t \in \mathbb{R}$ be such, that $a_{k+1}^{\downarrow} \leq t \leq a_k^{\downarrow}$, for some $1 \leq k \leq n$, then

$$\sum_{i=1}^{n} (a_i - t)^+ = \sum_{i=1}^{k} (a_i - t)^+ = \sum_{i=1}^{k} a_i^{\downarrow} - \sum_{i=1}^{k} t \leq \sum_{i=1}^{k} b_i^{\downarrow} - kt$$

$$= \sum_{i=1}^{k} (b_i^{\downarrow} - t) \leq \sum_{i=1}^{k} (b_i^{\downarrow} - t)^+ \leq \sum_{i=1}^{n} (b_i^{\downarrow} - t)^+ \Rightarrow \sum_{i=1}^{n} (a_i^{\downarrow} - t)^+ \leq \sum_{i=1}^{n} (b_i^{\downarrow} - t)^+.$$

Conversely, let

$$\sum_{i=1}^{n}(a_i^{\downarrow}-t)^{+} \leq \sum_{i=1}^{n}(b_i^{\downarrow}-t)^{+} \qquad (8)$$

To prove $a \prec_w b$, if $t = b_k^{\downarrow}$ for some $1 \leq k \leq n$,
Then

$$\sum_{i=1}^{n}(b_i-t)^{+} = \sum_{i=1}^{K}(b_i^{\downarrow}-t) = \sum_{i=1}^{K} b^{\downarrow}-kt \qquad (9)$$

But

$$\sum_{i=1}^{k} a_i^{\downarrow}-kt \leq \sum_{i=1}^{k}(a_i^{\downarrow}-t)^{+} = \sum_{i=1}^{n}(a_i-t)^{+} \qquad (10)$$

From the relations (8), (9) and (10) we get

$$\sum_{i=1}^{k} a_i^{\downarrow}-kt \leq \sum_{i=1}^{k} b_i^{\downarrow}-kt \Rightarrow \sum_{i=1}^{k} a_i^{\downarrow} \leq \sum_{i=1}^{k} b_i^{\downarrow} \Rightarrow a \prec_w b.$$

(2) Let $a \prec^w b$, this implies

$$\sum_{i=1}^{k} a_i^{\uparrow} \geq \sum_{i=1}^{k} b_i^{\uparrow},$$

To prove

$$\sum_{i=1}^{n}(t-a_i)^{+} \leq \sum_{i=1}^{n}(t-b_i)^{+}.$$

If
$a \prec^w b$, this implies that $-b \prec_w -a$.
By part (1), $-b \prec_w -a$ if and only if for every real number $t$,

$$\sum_{i=1}^{n}(-a_i+t)^{+} \leq \sum_{i=1}^{n}(-b_i+t)^{+},$$

Which is same as saying

$$\sum_{i=1}^{n}(t-a_i)^{+} \leq \sum_{i=1}^{n}(t-b_i)^{+}.$$

(3) Let $a \prec b$, which implies

$$\sum_{i=1}^{n} a_i^{\downarrow} = \sum_{i=1}^{n} b_i^{\downarrow} \text{ and } \sum_{i=1}^{k} a_i^{\downarrow} \leq \sum_{i=1}^{k} b_i^{\downarrow} \text{ for } 1 \leq k \leq n.$$

Also $a \prec b$ if and only if $a \prec_w b$ and $a \prec^w b$
by part (1) and (2) this holds if and only if

$$\sum_{i=1}^{n}(a_i-t)^{+} \leq \sum_{i=1}^{n}(b_i-t)^{+} \text{ and } \sum_{i=1}^{n}(t-a_i)^{+} \leq \sum_{i=1}^{n}(t-b_i)^{+},$$

That is if and only if

$$\sum_{i=1}^{n} |\, a_i - t| \leq \sum_{i=1}^{n} |\, b_i - t|.$$

**Majorization in Convex and Monotonic Functions**

In this section we give importance to the functions from $\mathbb{R}^n$ to $\mathbb{R}^m$, which preserve ordering majorization. Let $f: R \to \mathbb{R}$ be function, we will denote the map induced by $f$ on $\mathbb{R}^n$ also by $f$, that is

$$f(a) = \big(f(a_1) \dots f(a_n)\big)\ for\ a \in \mathbb{R}^n$$

Similarly the function $f: \mathbb{R}^n \to \mathbb{R}^m$ is convex if

$f(ta + (1-t)b) \leq tf(a) + (1-t)f(b)$ for $0 \leq t \leq 1$.

To show the function $|a - r| = f_r(a)$ is convex, let $a_1, a_2 \in \mathbb{R}^n$ and $0 \leq t \leq 1$,

$f_r(ta_1 + (1-t)a_2) = |ta_1 + (1-t)a_2 - r|$
$\leq |t||a_1 - r| + |(1-t)||a_2 - r|$
$= t|a_1 - r| + (1-t)|a_2 - r|$
$= tf(a_1) + (1-t)f(a_2)$, for $0 \leq t \leq 1$.
A useful characterization of majorization is the following:

**Theorem 3.4**

Let $a, b \in \mathbb{R}^n$, then the following two conditions are equivalent:

1. $a \prec b$.
2. $tr\, f(\mathrm{a}) \leq trf(\mathrm{b})$ for all convex functions $f$ from $R$ to $R$.

**Proof (1)** Let $a \prec b$, then $a = Ab$, for some doubly stochastic matrix $A$, so
$a_i = \sum_{j=1}^{n} a_{ij}\, b_j$
Where $a_{ij} \geq 0$ and $\sum_{j=1}^{n} a_{ij} = 1 \forall i$ Hence for every convex function $f$,

$$f(a_i) = f\left(\sum_{j=1}^{n} a_{ij}\, b_j\right) \leq \sum_{j=1}^{n} a_{ij}\, f(b_j).$$

Hence

$$\sum_{i=1}^{n} f\,(a_i) \leq \sum_{i,j}^{n} a_{ij}\, f(b_j) = \sum_{j=1}^{n} f\,(b_j) \Rightarrow \sum_{i=1}^{n} f\,(a_i) \leq \sum_{j=1}^{n} f\,(b_j)$$

That is for all convex functions $f$ from $R$ to $R$,

$tr\, f(\mathrm{a}) \leq trf(\mathrm{b})$
(2) Let $tr\, f(\mathrm{a}) \leq trf(\mathrm{b})$for all convex functions $f$ from $R$ to $R$, that is

$$\sum_{i=1}^{n} f\,(a_i) \leq \sum_{j=1}^{n} f\,(b_j).$$

Now let $f_r(a) = |a - r|$ and $f_r(b) = |b - r|$ be convex function, then we have

$$\sum_{i=1}^{n} |a_i - r| \leq \sum_{i=1}^{n} |b_i - r|$$

Which implies

$$\sum_{i=1}^{n} |\, a - r \Big| \leq \sum_{i=1}^{n} |\, b - r \Big|$$

$$\sum_{i=1}^{n}(a_i - r)^+ \leq \sum_{i=1}^{n}(b_i - r)^+ \qquad (11)$$

And

$$\sum_{i=1}^{n}(r - a_i)^+ \leq \sum_{i=1}^{n}(r - b_i)^+, \qquad (12)$$

Let us firstly consider

$$\sum_{i=1}^{n}(a_i - r)^+ \leq \sum_{i=1}^{n}(b_i - r)^+$$

And let $b_k^{\downarrow} = r$, then

$$\sum_{i=1}^{n}(b_i - r) \leq \sum_{i=1}^{k} b_i^{\downarrow} - kr \qquad (13)$$

But

$$\sum_{i=1}^{k} a_i^{\downarrow} - kr \leq \sum_{i=1}^{n}(a_i - r)^+ \qquad (14)$$

From (11), (13) and (14) we get

$$\sum_{i=1}^{k} a_i^{\downarrow} - kr \leq \sum_{i=1}^{k} b_i^{\downarrow} - kr \Rightarrow a \prec_w b. \qquad (15)$$

Now consider (12), that is

$$\sum_{i=1}^{n}(r - a_i)^+ \leq \sum_{i=1}^{n}(r - b_i)^+$$

Let $r = b_k^{\downarrow}$, then we have

$$\sum_{i=1}^{n}(r - b_i^{\downarrow})^+ \geq \sum_{i=1}^{k}(r - b_i^{\downarrow}) = kr - \sum_{i=1}^{k} b_i^{\downarrow}, \text{ but } kr - \sum_{i=1}^{k} a_i^{\uparrow} \geq \sum_{i=1}^{n}(r - a_i^{\uparrow})^+.$$

For the following inequality

$$\sum_{i=1}^{n}(r - a_i^{\uparrow})^+ \leq \sum_{i=1}^{n}(r - b_i^{\uparrow})^+,$$

We must have

$$kr - \sum_{i=1}^{k} a_i^{\uparrow} \leq kr - \sum_{i=1}^{k} b_i^{\uparrow} \Rightarrow a \overset{w}{\prec} b. \qquad (16)$$

Now from (15) and (16) we get $a \prec b$.
Next we consider majorization on monotonic functions.

**Theorem 3.5**

For $a, b \in \mathbb{R}^n$, the following two conditions are equivalent:

1. $a \prec_w b$.
2. $tr\, f(\mathrm{a}) \leq tr f(\mathrm{b})$ for all monotonically increasing convex functions $f$ from $\mathbb{R}$ to $\mathbb{R}$.

**Proof:**

Let

$a \prec_w b$, for $a, b \in \mathbb{R}^n$, that is $\sum_{i=1}^{k} a_i^{\downarrow} \leq \sum_{i=1}^{k} b_i^{\downarrow}$.

Consider a function $f_r(a) = (a - r)$, firstly we will show that this function is convex.

$$f_r(ta_1 + (1-t)a_2)$$
$$\leq (t(a_1 - r) + (1-t)(a_2 - r))^+$$
$$= tf(a_1) + (1-t)f(a_2) \text{ for } 0 \leq t \leq 1$$

$\Rightarrow f_r(a)$ is convex.

To show $f_r(a)$ is monotonically increasing function, that is if $a_1 \leq a_2$, then $f_r(a_1) \leq f_r(a_2)$:

$$f_r(a_1) = (a_1 - r) \leq (a_1 - r)^+ = a_1 - r$$
$$f_r(a_2) = (a_2 - r) \leq (a_2 - r)^+ = a_2 - r$$
$$\Rightarrow f_r(a_1) \leq f_r(a_2).$$

Let $a \in \mathbb{R}^n$, be in decreasing order. Consider $r = a_k$, then $(a_i^{\downarrow} - r)^+ = 0 \forall i \geq k$

$$tr f_r(a) = \sum_{i=1}^{n} (a_i - r) \leq \sum_{i=1}^{k} (a_i^{\downarrow} - r)^+$$

$$= \sum_{i=1}^{k} a_i^{\downarrow} - kr \leq \sum_{i=1}^{k} b_i^{\downarrow} - kr, \text{ for } r = b_k$$

$$\leq \sum_{i=1}^{n} (b_i^{\downarrow} - r)$$

$\Rightarrow tr\, f_r(\mathrm{a}) \leq tr f_r(\mathrm{b})$.

Conversely, let

$tr\, f_r(\mathrm{a}) \leq tr f_r(\mathrm{b})$ For $a, b \in \mathbb{R}^n$ (17)

To prove $a \prec_w b$, let $a_{k+1} \leq r \leq a_k$

$$tr\, f_r(a) = \sum_{i=1}^{n} (a_i - r) \leq \sum_{i=1}^{k} (a_i^{\downarrow} - r)^+ \ , \ \text{ for } 0 \leq k \leq n$$

$$= \sum_{i=1}^{k} a_i^{\downarrow} - kr, \qquad (18)$$

But

$$\sum_{i=1}^{k} b_i^{\downarrow} - kr \leq \sum_{i=1}^{k} (b_i^{\downarrow} - r)^+, \text{ for } r = b_k$$

$$\leq \sum_{i=1}^{n} (b_i^{\downarrow} - r)^+ = \sum_{i=1}^{n} (b_i^{\downarrow} - r)\,, \qquad (19)$$

From (17), (18) and (19) we get

$$\sum_{i=1}^{k} a_i^{\downarrow} \leq \sum_{i=1}^{k} b_i^{\downarrow} \Rightarrow a \prec_w b$$

Furthermore, a real valued function $f$ on $\mathbb{R}^n$ satisfying $a \prec b \Rightarrow f(a) \leq f(b)$ is Schur-convex, which expresses preservation of order rather then convexity.

Let $f: \mathbb{R}^n \to \mathbb{R}^m$, the domain of $f$ is either all of $\mathbb{R}^n$, or some convex set invariant under co-ordinate permutations of its elements. Such a map is monotonically increasing if

Monotonically decreasing if $-f$ is monotonically increasing,

$$a \prec b \Rightarrow f(a) \leq f(b)$$

Also it is convex if

$f(ta + (1-t)b) \leq tf(b) + (1-t)f(b)\ 0 \leq t \leq 1$

And concave if $-f$ is convex.

**Theorem 3.6**

Let $f: \mathbb{R}^n \to \mathbb{R}^m$ be a convex map, suppose it, for any $P \in S_n$, then there exist $P' \in S_m$ such that

$$f(Pa) = P'f(a), \quad \forall a \in \mathbb{R}^n, \qquad (20)$$

Where $S_n$ denotes the grope of $n \times n$ permutation matrix. In addition if $f$ is monotonically increasing, then $f$ is strongly isotone.

**Proof 1**

Let $a \prec b$ in $\mathbb{R}^n$, by the lemma (2.9) there exists $P_1, P_2, \ldots, P_N \in S_n$ permutation matrices and real positive numbers $\lambda_1, \lambda_2, \ldots, \lambda_N$ with $\sum_{j=1}^{n} \lambda_j = 1$, such that

$$a = \sum_{j=1}^{n} \lambda_j P_j b.$$

So by (20) and convexity of $f$ we have

$$f(a) = f\left(\sum_{j=1}^{n} \lambda_j p_j b\right) \leq \sum_{j=1}^{n} \lambda_j f(p_j b)$$

$$= \sum_{j=1}^{n} \lambda_j p'_j f(b) = c \text{ say },$$

Then

$$c \prec f(b) \text{ and } f(a) \leq c \quad \text{so } f(a) \prec_w f(b).$$

Now suppose $f$ is monotonically increasing, let $d \prec_w b$ then, by (2.10) there exists $a$, such that $d \leq a \prec b$, hence $f(d) \leq f(a)$ and $f(a) \prec_w f(b)$ so $f(d) \prec_w f(b)$.

Recall the remark from the literature (Bhatia, 1997)as follows.

1. If $f: \mathbb{R} \to \mathbb{R}$ is convex, then the induced map $f: \mathbb{R}^n \to \mathbb{R}^n$ is isotone.
2. If $f: \mathbb{R} \to \mathbb{R}$ is convex and monotonically increasing, then the induced map $f: \mathbb{R}^n \to \mathbb{R}^n$ is strongly isotone.

The above results imply that:

1. $a \prec b$ in $\mathbb{R}^n \Rightarrow |a| \prec_w |b|$.
2. $a \prec b$ in $\mathbb{R}^n \Rightarrow a^2 \prec_w b^2$.
3. $a \prec_w b$ in $\mathbb{R}^n_+ \Rightarrow a^p \prec_w b^p$ for $p > 1$.
4. $a \prec_w b$ in $\mathbb{R}^n \Rightarrow a^+ \prec_w b^+$.

$\mathbb{R}_+^n$ Here stands for the collection of vectors $a \geq 0$, $|a| = (|a_1|, |a_2|, \ldots, |a_n|)$ and $a^+ = a \vee 0$.

**Proof 1**

Let $a \prec b$ in $\mathbb{R}^n$, and let $f(a) = |a|$ we want to show that $f(a) = |a|$ is convex in $\mathbb{R}$.

$f(a) = |a|$, for some $0 \leq t \leq 1$

$f(ta + (1-t)b) \leq |t|a| + (1-t)|b|$

$= tf(a) + (1-t)f(b)$, for $0 \leq t \leq 1$

So by first result the function $f(a) = |a|$ is isotone in $\mathbb{R}^n$.

(2) Let $a \prec b$ in $\mathbb{R}^n$, and let $f(a) = a^2$ we want to show that $f(a) = a^2$ is convex in $\mathbb{R}$,

$$f(ta + (1-t)b) = t^2a^2 + (1-t)^2b^2 + 2ta(1-t)b$$
$$\leq \quad ta^2 + (1-t)b^2$$
$$= \quad tf(a) + (1-t)f(b) \text{ for } 0 \leq t \leq 1$$

Then, by first result the function $f(a) = a^2$ is isotone in $\mathbb{R}^n$.

(3) Let $a \prec_w b$ in $\mathbb{R}_+^n$, and let $f(a) = a^p$, we want to show that $f(a) = a^p$ is convex and monotonically increasing in $\mathbb{R}_+$.

To show the function $f(a) = a^p$ is convex in $R_+$ for $p > 1$,

$f''(a) \geq 0$, since $p > 1$, so it is convex.

Let $a \leq b \Rightarrow a^p \leq b^p$, since $a, b \in R_+$ and $P > 1$, so it is monotonically increasing in $\mathbb{R}_+$.

This implies that, for $p > 1 \Rightarrow f(a) = a^p$ is strongly isotone in $\mathbb{R}_+^n$, so $a^p \prec_w b^p$.

(4) Let $f(a) = a^+$, and $a \prec_w b$ in $\mathbb{R}^n \Rightarrow \sum_{i=1}^k a_i^\downarrow = \sum_{i=1}^k b_i^\downarrow$

Since $a^+$ is obtained from $a$ by replacing the negative co-ordinates of $a$, by 0 so if there is negative co-ordinates in $a$, then

$$\sum_{i=1}^k a_i^\downarrow \leq \sum_{i=1}^k (a_i^\downarrow)^+.$$

Similarly for $b$

$$\sum_{i=1}^k b_i^\downarrow \leq \sum_{i=1}^k (b_i^\downarrow)^+ \Rightarrow \sum_{i=1}^k a_i^\downarrow \leq \sum_{i=1}^k (a_i^\downarrow)^+ \leq \sum_{i=1}^k b_i^\downarrow \leq \sum_{i=1}^k (b_i^\downarrow)^+$$

$$\Rightarrow \sum_{i=1}^k (a_i^\downarrow)^+ \leq \sum_{i=1}^k (b_i^\downarrow)^+, \qquad \forall 1 \leq k \leq n$$

$$\Rightarrow a^+ \prec_w b^+.$$

Returning to Theorem (3.6) we note that for $m = 1$, the condition (20), that is

$[f(pa) = p'f(a) \forall a \in \mathbb{R}^n]$,

Says $f$ is permutation invariant, that is

$f(pa) = f(a) \forall a \in \mathbb{R}^n$ and $p \in S_n$.

In this case Theorem (3.6) says that if a function $f: \mathbb{R}^n \to \mathbb{R}$ is convex and permutation invariant, then it is isotone.

Also every isotone function $f: \mathbb{R}^n \to \mathbb{R}$ has to be permutation invariant, because $pa$, and $a$, majorized each other, that is $pa \prec a$, hence being isotone of $f$ implies $f(pa) = f(a)$ in this case.

**Theorem 3.7**

Let $f: \mathbb{R}^n \to \mathbb{R}$ be a convex function and let.

$$g(a) = \max_{P \in S_n} f(Pa),$$

Then $g$ is isotone, if in addition $f$ is monotonically increasing, then $g$ is strongly isotone.

**Proof 1**

Let $P' \in S_n$ be permutation matrix,

$$g(P'a) = \max_{p \in S_n} f(PP'a).$$

Now $PP'$ is again a permutation matrix so we can write $PP'$=$Q$, so $PP'$ $(a)$=$Q(a)$. Since $Q$ is

$$g(P'a) = \max_{Q \in S_n} f(Qa),$$

Which implies $g$ is convex.

$$f(ta + (1-t)b) = \max_{Q \in S_n} f\,[tP(b) + (1-t)Pb]$$

$$\leq \max_{p \in S_n}[tf(Pa) + (1-t)f(Pb)]$$

$$= tg(a) + (1-t)g(b) \Rightarrow g \text{ is convex}$$

So we can conclude that, $g$ is isotone.

Let $a \leq b$

$$\Rightarrow Pa \leq Pb, \qquad \forall\, permutaton\, P.$$

$$f(Pa) \leq f(Pb).$$

$$\max_{p \in S_n} f(Pa) \leq \max_{p \in S_n} f(Pb) \Rightarrow g(Pa) \leq g(Pb),$$

So, $g$ is monotonically increasing, hence $g$ is strongly isotone.

## CONCLUSION:

There are many conditions on vectors which implies Majorization. Some of these conditions presents weakly submajorization and some shows weakly supermajorization which are forms of majorization, we defined all of them in this paper. The properties of Majorization, Weakly supermajorization and weakly submajorization and the relations between them are explored in this work with examples. By considering the concept of our title Majorization and its applications on some Functions, we summarized the application of Majorization on some functions like monotonic functions, convex functions and so on with some properties. Such results are explained by theorems and examples.

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## CONFLICTS OF INTEREST:

All authors have contributed to this research, and publishing it has no potential conflicts of interest.

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