Publisher homepage: www.universepg.com, ISSN: 2707-4625 (Online) & 2707-4617 (Print)

https://doi.org/10.34104/ijmms.023.048061

International Journal of Material and Mathematical Sciences

Journal homepage: www.universepg.com/journal/ijmms

Similar Solutions for Changing Characteristics of Natural Convection Laminar Flow around a Vertical Rectangular Inclined Plane Surface

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ABSTRACT

The study investigates laminar natural convection flow around a vertical rectangular inclined plane surface, specifically focusing on the effects of viscous and pressure stress work. Previous research primarily addressed these effects in two-dimensional natural convection flow scenarios, while this study extends the analysis to encompass three exterior situations, accounting for variations in fluid properties outside compressible boundary layers. Employing numerical methods, the study aims to predict essential flow parameters including Prandtl's number (Pr), controlling parameter (C), and additional parameters (A, B, D, E, ∅). The investigation yields velocity profiles (F'(0), S'(0)), temperature profiles $\theta(0)$, skin frictions (F''(0), S''(0)), and heat transfer coefficients θ'(0). Numerical results are presented to illustrate the impact of different parameters on these flow characteristics. Additionally, tabulated data in Table 2 and Table 3 offer further insights into the predicted flow parameters and their variations under varying conditions.

Keywords: Natural Convection, Inclined Plane, Heat Transfer, Viscous Dissipation, Pressure Work.

INTRODUCTION:

Howarth, (1953) discussed the properties of these solutions for $c=\frac{b}{a}=0, \frac{1}{4}, \frac{1}{2}, \frac{3}{4}$, and 1. The limiting values c=0 and 1correspond to the two-dimensional and axially symmetric stagnation points, respectively. Later Davey (1961) showed that similar solutions exist for $c \geq -1$. The similar solutions for $-1 \leq c < o$ correspond to the flow near saddle-points of attachment and may in some cases are related to the flow in the vicinity of geometrical saddle-points on the surface (Akter *et al*., 2023).

The energy equation for an elementary fluid volume was derived by Howarth, (1953) through the balance of the rates of convection of particular internal and kinetic energies into the volume, as well as the rates of heat conduction into the volume and the rates of body force, pressure, and viscous stresses acting on the volume. This concept leads to a number of alternate variants of the energy equation, which are found by substituting some other feature of state for internal energy. Moreover, the momentum equation is typically used to eliminate the kinetic energy element. It is important to remember that only specific phrases that indicate the speeds at which pressure and viscous stresses operate are still expressly present in all of these different formulations.

These types of pressure and viscous stress work effects are typically disregarded in discussions and analyses of natural convection flows. However (Gebhart, 1962; Gebhart and Mollendorf, 1962) have also looked at the significance and impact of viscous stress work effects in laminar flows. Special flows over semi-infinite flat surfaces oriented parallel to the direction of body force were taken into consideration in each of these investigations. Gebhart, (1962) considered flows generated by the plate surface temperatures which vary as powers of ξ (the distance along the plate surface from the leading edge), and Mollendorf, (1969) considered flows generated by plate surface temperatures which vary exponentially in ξ. In both of these investigations the length scale was $\frac{C_p}{G\beta}$, where β is the volumetric co-efficient of thermal expansion. Since this length scale is usually extremely large for most fluids, it was shown that viscous stress work effects are very small in most situations. For example, in the case of constant surface temperature, Gebhart, (1962) showed that viscous stress work effects are governed simply by the ratio of ξ and the above large length scale. It is easy to show that this ratio is the Eckert number for this flow.

Free convection from a vertical permeable circular cone with pressure work and non-uniform surface temperature was carried out by Alam *et al.* (2007). The impact of viscous dissipation and pressure stress work in natural convection flow along a vertical flat plate with heat conduction was examined by Alam *et al.* (2006). The Joule heating effect on the coupling of conduction with magneto-hydrodynamic free convection flow from a vertical flat plate was investigated by Alim *et al.* (2007). The joint impact of joule heating and viscous dissipation on the coupling of conduction and free convection down a vertical flat plate was illustrated by Alim *et al.* (2008). The numerical investigation of temperature-dependent viscosity and thermal conductivity on a natural convection flow across a sphere in the presence of magneto-hydrodynamics was covered by Alam *et al.* (2018). Viscos dissipation and temperature-dependent viscosity on MHD free convection flow over a sphere with heat conduction were numerically studied by Alam *et al.* (2018). It is noted that several studies have focused on including both pressure work effects and viscous dissipation in the energy equation for laminar boundary layer natural convection flows. Conventional analysis typically overlooks both the effects. Here, the energy equation retains the effects of pressure and viscous stress work. The effects of pressure and viscous stress work on natural convection flow on a flat surface were investigated by Ackroyd, (1974). In this instance, we also looked at how pressure and viscous stress interact to create laminar natural convection flow around a plane surface that is inclined and vertical.

Boundary-layer equations and transformations

We examine a three-dimensional laminar natural convection fluid flow that is stable and has a high Reynolds number around an OABC, which is a vertical, inclined plane surface. The selection of cartesian coordinates (ξ,η,ζ) is made according to fig. A. While stretches into the boundary layer, the co-ordinates ξ and η are thought to lie and be defined in the surface that the boundary layer is flowing over. Here ζ represents an actual distance from the surface measured along a straight normal in this instance. Where the gravitational force $-\vec{G}=\left(-\vec{g}_{\xi},-\vec{g}_{\eta},0\right)$ is, the components of the gravity vector are g_{ξ} and g_{η}, respectively, in the ξ and η directions.

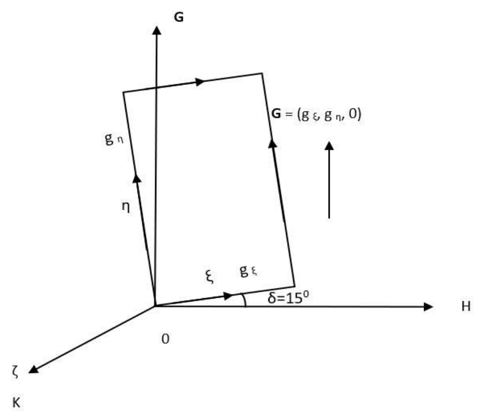

Fig. 1: Geometry.

OH direction is the horizontal direction. *OG* is the vertical direction. *OK* is ⊥ to the plane. ξOη rectangular plane is inclined at angle δ with vertical plane *HOG*.

The compressible laminar natural convection flow about a vertical rectangular inclined surface is governed by the following equations: -

Continuity equation, $\frac{\partial}{\partial\xi}(\rho u)+\frac{\partial}{\partial\eta}(\rho v)+\frac{\partial}{\partial\zeta}(\rho w)=0$ (1)

u- momentum equation, $\rho\left(u\frac{\partial u}{\partial \xi}+v\frac{\partial u}{\partial \eta}+w\frac{\partial u}{\partial \zeta}\right)=-\frac{\partial p}{\partial \xi}+\rho \quad g_{\xi}+\frac{\partial}{\partial \zeta}\left(\mu\frac{\partial u}{\partial \zeta}\right)$ (2)

v- momentum equation, $\rho\left(u\frac{\partial v}{\partial \xi}+v\frac{\partial v}{\partial \eta}+w\frac{\partial v}{\partial \zeta}\right)=-\frac{\partial p}{\partial \eta}+\rho \quad g_{\eta}+\frac{\partial}{\partial \zeta}\left(\mu\frac{\partial v}{\partial \zeta}\right)$ (3)

and w- momentum equation, $\frac{\partial w}{\partial \zeta}=0$ (4)

The energy equation in simplified form

$$\rho \quad C_p\left(u\frac{\partial T}{\partial \xi}+v\frac{\partial T}{\partial \eta}+w\frac{\partial T}{\partial \zeta}\right)-\left(u\frac{\partial p}{\partial \xi}+v\frac{\partial p}{\partial \eta}\right)=\frac{\partial}{\partial \zeta}\left(k\frac{\partial \theta}{\partial \zeta}\right)+\mu\left\{\left(\frac{\partial u}{\partial \zeta}\right)^2+\left(\frac{\partial v}{\partial \zeta}\right)^2\right\} \tag{5}$$

The terms $\rho \quad g_{\xi}, \rho \quad g_{\eta}$ represent the body force components exerted on fluid particle.

For thermally perfect gas , $\beta=\frac{1}{T}$ so that $\beta T=1$ and $\beta=-\frac{1}{\rho}\left(\frac{\partial p}{\partial T}\right)_p$ (6)

Conditions exterior to the boundary layer (denoted by suffix e), at which the exterior fluid is at rest

$$\left.\begin{array}{l}\text{for thermally perfect gas} \quad \rho_e g_{\xi}-\frac{dp}{d\xi}=0 \quad \text{implies that} \quad \rho_e g_{\xi}=\frac{dp}{d\xi}\\ \text{and} \quad \rho_e g_{\eta}-\frac{dp}{d\eta}=0 \quad \text{implies that} \quad \rho_e g_{\eta}=\frac{dp}{d\eta}\end{array}\right\} \tag{7}$$

Therefore, the modified three-dimensional governing equations for the compressible steady flows [using (3-6e) and (3-7)] are,

Continuity equation $\frac{\partial}{\partial \xi}(\rho u)+\frac{\partial}{\partial \eta}(\rho v)+\frac{\partial}{\partial \zeta}(\rho w)=0$ (8)

u-momentum and v-momentum equations: -

$$\rho\left(u\frac{\partial u}{\partial \xi}+v\frac{\partial u}{\partial \eta}+w\frac{\partial u}{\partial \zeta}\right)=\left(\rho-\rho_e\right)g_{\eta}+\frac{\partial}{\partial \zeta}\left(\mu\frac{\partial u}{\partial \zeta}\right) \tag{9}$$

and $\rho\left(u\frac{\partial v}{\partial \xi}+v\frac{\partial v}{\partial \eta}+w\frac{\partial v}{\partial \zeta}\right)=\left(\rho-\rho_e\right)g_{\eta}+\frac{\partial}{\partial \zeta}\left(\mu\frac{\partial v}{\partial \zeta}\right)$ (10)

Also the energy equations, $\rho C_p\left(u\frac{\partial T}{\partial \xi}+v\frac{\partial T}{\partial \eta}+w\frac{\partial T}{\partial \zeta}\right)-T\beta\left(u\frac{\partial p}{\partial \xi}+v\frac{\partial p}{\partial \eta}\right)$

$$=\frac{\partial}{\partial \zeta}\left(k\frac{\partial T}{\partial \zeta}\right)+\mu\left[\left(\frac{\partial u}{\partial \zeta}\right)^2+\left(\frac{\partial v}{\partial \zeta}\right)^2\right] \tag{11}$$

where $\phi=\mu\left[\left(\frac{\partial u}{\partial \zeta}\right)^2+\left(\frac{\partial v}{\partial \zeta}\right)^2\right]$ is known as the viscous dissipation function and ϕ represents that part of the viscous work necessary for the correct balance of energy in this particular form of the energy equation, the term $T\beta\left[u\frac{\partial p}{\partial \xi}+v\frac{\partial p}{\partial \eta}\right]$ represents the corresponding part of the pressure work. It is the latter term which is ignored by Gebhart, (1962) and Gebhart and Mollendorf, (1962). Ackroyd, (1974) and Zakerullah, (1962) considered for two-dimensional and axi-symmetric flow respectively status reasons for their inclusion simultaneously.

Now the non-dimensional temperature difference,

$$\theta(\xi,\eta,\zeta)=\frac{T(\xi,\eta,\zeta)-T_e(\xi,\eta)}{T_w(\xi,\eta)-T_e(\xi,\eta)}=\frac{T(\xi,\eta,\zeta)-T_e(\xi,\eta)}{\Delta T(\xi,\eta)} \tag{12}$$

Where

$$\Delta T(\xi,\eta)=T_w(\xi,\eta)-T_e(\xi,\eta) \tag{13}$$

We change the independent variables (ξ,η,ζ) to a new set of variables (*X*, *Y*, Φ) and relations between two sets of variable are given by:

$$\xi = X,\ \eta = Y \quad \text{and} \quad \Phi = \left[\frac{\rho_r^2\left(-G(\Delta T)_r\right)\beta_r}{16\mu_r^2(X+CY)}\right]^{\frac{1}{4}} \int_0^{\zeta} \frac{\rho_r}{\rho}\, d\zeta_1 \tag{14}$$

C is the additive parameter and the suffix '*r*' refers to any convenient reference condition and also $(\Delta T)_r$ is any convenient reference temperature difference.

Let two stream functions $\Psi(\xi,\eta,\zeta)$ and $\chi(\xi,\eta,\zeta)$ be defined as the mass flow components within the boundary layer for the case of compressible viscous flow. To satisfy the equation of continuity, we may introduce the components of the mass flow in the following way,

$$\rho u = \frac{\partial \psi}{\partial \zeta},\ \rho v = \frac{\partial \chi}{\partial \zeta} \quad \text{and} \quad \rho w = -\left(\psi_\xi + \chi_\eta\right)\Big\} \tag{15}$$

In order to seek the similarity functions, we introduce the following equations,

$$\int_0^{\Phi} \frac{\rho\ \ u}{\rho_r U_F}\, d\Phi = F(X,Y,\Phi) \quad \text{and} \quad \int_0^{\Phi} \frac{\rho\ \ v}{\rho_r U_F}\, d\Phi = S(X,Y,\Phi) \tag{16}$$

Where $U_F = -\sqrt{G(\Delta T)_r\,\beta_r\,(X+CY)}$ which represent the characteristic velocity (maximum) generated by the buoyancy effect and (*X+CY*) denotes some characteristic length. Therefore,

$$\frac{\rho u}{\rho_r U_F} = F_\Phi(X,Y,\Phi) \quad \text{and} \quad \frac{\rho\ \ v}{\rho_r U_F} = S_\Phi(X,Y,\Phi) \tag{17}$$

Thus we have $$\psi(X,Y,\Phi) = \left[16\mu_r^2\rho_r^2\left(-G(\Delta T)_r\right)\beta_r(X+CY)^3\right]^{\frac{1}{4}}\left(\frac{\rho_r}{\rho}F(X,Y,\Phi)\right) \tag{18}$$

Similarly we have, $$\chi(X,Y,\Phi) = \left[16\mu_r^2\rho_r^2\left(-G(\Delta T)_r\right)\beta_r(X+CY)^3\right]^{\frac{1}{4}}\left(\frac{\rho_r}{\rho}S(X,Y,\Phi)\right) \tag{19}$$

Equations (9), (10) and (11) can be written most simplified form using the above relations, we have

u-Momentum Equation:

$$\begin{aligned}
&\left\{\frac{\rho\mu}{\rho_r\mu_r}\left(\frac{\rho_r}{\rho}F\right)''\right\}' + \frac{4\left(\frac{\rho_e}{\rho}-1\right)g_\xi}{G(\Delta T)_r\,\beta_r} - 2\left(\frac{\rho_r}{\rho}F\right)'^2 - 2C\left(\frac{\rho_r}{\rho}S\right)'\left(\frac{\rho_r}{\rho}F\right)' \\
&-\left(\frac{\rho_r}{\rho}F\right)''\frac{\partial}{\partial X}\left(\frac{\rho_r}{\rho}F\right) + 3\left(\frac{\rho_r}{\rho}F\right)\left(\frac{\rho_r}{\rho}F\right)'' = 4(X+CY)\Bigg[\left\{\left(\frac{\rho_r}{\rho}F\right)'\frac{\partial}{\partial X}\left(\frac{\rho_r}{\rho}F\right)'\right\} \\
&-\left(\frac{\rho_r}{\rho}F\right)''\frac{\partial}{\partial X}\left(\frac{\rho_r}{\rho}F\right) + \left\{\left(\frac{\rho_r}{\rho}S\right)'\frac{\partial}{\partial Y}\left(\frac{\rho_r}{\rho}F\right)' - C\left(\frac{\rho_r}{\rho}F\right)''\frac{\partial}{\partial Y}\left(\frac{\rho_r}{\rho}S\right)\right\}\Bigg]
\end{aligned} \tag{20}$$

v-Momentum Equation

$$\begin{aligned}
&\left\{\frac{\rho\mu}{\rho_r\mu_r}\left(\frac{\rho_r}{\rho}S\right)''\right\}' + \frac{4\left(\frac{\rho_e}{\rho}-1\right)g_\eta}{G(\Delta T)_r\,\beta_r} - 2C\left(\frac{\rho_r}{\rho}S\right)'^2 - 2\left(\frac{\rho_r}{\rho}S\right)'\left(\frac{\rho_r}{\rho}F\right)' \\
&-\left(\frac{\rho_r}{\rho}S\right)''\frac{\partial}{\partial X}\left(\frac{\rho_r}{\rho}F\right) + 3\left(\frac{\rho_r}{\rho}F\right)\left(\frac{\rho_r}{\rho}S\right)'' = 4(X+CY)\Bigg[\left\{\left(\frac{\rho_r}{\rho}F\right)'\frac{\partial}{\partial X}\left(\frac{\rho_r}{\rho}S\right)'\right\} \\
&-\left(\frac{\rho_r}{\rho}S\right)''\frac{\partial}{\partial X}\left(\frac{\rho_r}{\rho}F\right) + \left\{\left(\frac{\rho_r}{\rho}S\right)'\frac{\partial}{\partial Y}\left(\frac{\rho_r}{\rho}S\right)' - C\left(\frac{\rho_r}{\rho}S\right)''\frac{\partial}{\partial Y}\left(\frac{\rho_r}{\rho}S\right)\right\}\Bigg]
\end{aligned} \tag{21}$$

And the Energy Equation

$$\frac{C_{p_r}}{C_p}\left(\frac{\rho\mu}{\rho_r\mu_r}\frac{C_p}{C_{p_r}}\frac{\theta'}{P_r}\right)'+3\left\{\left(\frac{\rho_r}{\rho}F\right)+C\left(\frac{\rho_r}{\rho}S\right)\right\}\theta'-\frac{4(X+CY)}{\Delta T}\left(\frac{\partial T_e}{\partial X}+\theta\frac{\partial \Delta T}{\partial X}\right)\left(\frac{\rho_r}{\rho}F\right)'$$

$$-\frac{4(X+CY)}{\Delta T}\left(\frac{\partial T_e}{\partial Y}+\theta\frac{\partial \Delta T}{\partial Y}\right)\left(\frac{\rho_r}{\rho}S\right)'+\frac{(X+CY)C_{p_r}}{LC_p}\left[\left[\left\{\frac{\rho\mu}{\rho_r\mu_r}\frac{(\Delta T)_r}{\Delta T}\left(\left(\frac{\rho_r}{\rho}F\right)''^2+\left(\frac{\rho_r}{\rho}S\right)''^2\right)\right\}\right.\right.$$

$$\left.-\frac{4g_\xi}{G}\frac{\rho_e}{\rho}\frac{T\beta}{\beta_r\Delta T}\left(\frac{\rho_r}{\rho}F\right)'-\frac{4g_\eta}{G}\frac{\rho_e}{\rho}\frac{T\beta}{\beta_r\Delta T}\left(\frac{\rho_r}{\rho}S\right)'\right]=4(X+CY)\left[\left[\left\{\frac{\partial\theta}{\partial X}\left(\frac{\rho_r}{\rho}F\right)'-\frac{\partial}{\partial X}\left(\frac{\rho_r}{\rho}F\right)\theta'\right\}\right.\right.$$

$$\left.\left.+\left\{\frac{\partial\theta}{\partial Y}\left(\frac{\rho_r}{\rho}S\right)'-\frac{\partial}{\partial Y}\left(\frac{\rho_r}{\rho}S\right)\theta'\right\}\right]\right] \tag{22}$$

Where $F(X,Y,\Phi)=F$, $S(X,Y,\Phi)=S$ and $T(X,Y,\Phi)=T$ are written respectively.

Also, the boundary conditions of the above equations are

$$\left.\begin{array}{l} F(\xi,\eta,0)=F'(\xi,\eta,0)=F'(\xi,\eta,\infty)=0,\ S(\xi,\eta,0)=S'(\xi,\eta,0)=S'(\xi,\eta,\infty)=0 \\ \text{and} \qquad \theta(\xi,\eta,0)=1,\ \theta(\xi,\eta,\infty)=0 \end{array}\right\} \tag{23}$$

Here primes denote differentiation with respect to Φ and the parameter L (which has the dimensions of length) is defined as

$$L=\frac{C_{p_r}}{-G\beta_r} \tag{24}$$

In left hand side of (22), the effect of viscous work and pressure work are found to be proportional to the first and second, third terms in square brackets respectively.

The first term is of order $\left(\frac{\rho_r}{\rho}F\right)''^2+\left(\frac{\rho_r}{\rho}S\right)''^2$,

The 2nd and 3rd terms are of order $\left(\frac{\rho_r}{\rho}F\right)'\frac{T_r}{(\Delta T)_r}$ and $\left(\frac{\rho_r}{\rho}S\right)'\frac{T_r}{(\Delta T)_r}$ respectively. It would appear that for both liquids and gases the effect of pressure work is not necessarily small in comparison with the effect of viscous work. Furthermore, both stress work terms are seen to be multiplied by $\frac{X+CY}{L}$. Thus, we note that the importance of the two stress work terms (relative to those terms representing convection and diffusion of heat) is determined largely by the nature of the T_e and ΔT variation. These points will be examined in detail in the following sections.

Now (14) can be written as,

$$\Phi=\frac{1}{2}\frac{R_e^{1/2}}{(X+CY)}\int_0^\zeta\frac{\rho}{\rho_r}d\zeta_1 \tag{25}$$

Here $R_e=\frac{\rho_r U_F(X+CY)}{\mu_r}$ is a Reynolds number based on the characteristic free convection velocity U_F defined by Ostrach (1964), where ,

$$U_F^2=-G\beta_r(\Delta T)_r(X+CY) \text{ and } \frac{X+CY}{L}=\frac{U_F^2}{C_{P_r}(\Delta T)_r} \tag{26}$$

Which is the Eckert number based on U_F and characteristic length (X+CY).

We shall obtain solutions of (20), (21) and (22) to first order of $\frac{X+CY}{L}$ only, because is the most cases of natural convection flow $(X+CY)/L$<<1as explained by Ackroyed, (1962). condition. For thermally perfect gas,

$$p\propto\rho T,\ \beta=\frac{1}{T} \tag{27}$$

Following procedure as Ackroyd, (1974) we have

$$\frac{\frac{\rho_e}{\rho}-1}{\beta_r(\Delta T)_r}=\theta\frac{\Delta T}{(\Delta T)_r}\frac{T_r}{T_e}\ ,\ \frac{T}{\Delta T}\frac{\rho_e}{\rho}\frac{\beta}{\beta_r}=\frac{T_r}{\Delta T}+\theta\frac{T_r}{T_e} \tag{28}$$

$$d\rho=-\rho\beta_T dT+\rho \mathrm{K} dp\ \textbf{ or, }\frac{d\rho}{\rho}=-\beta_T dT+\mathrm{K}dp\ \text{ or, }\frac{\rho_e}{\rho_r}=e^{-\left\{\beta_T T_r T'-\mathrm{K}p_r p'\right\}}$$

i.e $\quad \frac{\rho_e}{\rho_r}\simeq 1-\beta_r T_r T'+\mathrm{K}_r p_r p'$ (29)

where $\beta_r=-\left\{\frac{1}{\rho}\left(\frac{\partial\rho}{\partial T}\right)_p\right\}_r\ ,\ \mathrm{K}_r=\left\{\frac{1}{\rho}\left(\frac{\partial\rho}{\partial p}\right)_T\right\}_r$

and $T'=\frac{T_e-T_r}{T_r},\ p'=\frac{p_e-p_r}{p_r},\ \frac{T-T_r}{T_r}=T'=\frac{\Delta T}{T_r}\theta+T'$ (30)

Where T', p' and $\frac{\Delta T}{T_r}$ are all small compared with unity, and θ is at most of order unity in the boundary layer but $\theta\equiv 0$ in the exterior fluid, we shall find in the following section that the leading terms in the expressions for T and p' are of order $\frac{X+CY}{L}$. Again using (27), also we have -

$$\frac{\rho_e}{\rho}=\frac{\rho_e}{\rho_r}\frac{\rho_r}{\rho}\simeq\left(1-\beta_r T_r T'+\mathrm{K}_r p_r p'\right)\left\{1+\beta_r T_r T'-\mathrm{K}_r p_r p'+\frac{1}{2}\left(\mathrm{K}^2-\left(\frac{\partial \mathrm{K}}{\partial T}\right)_p\right)_r p_r^2 p'^2\right.$$
$$+\frac{\Delta T}{T_r}\theta\left[\beta_r T_r+\frac{\Delta T}{T_r}\theta\frac{\beta_r^2T_r^2}{2}\left\{1+\frac{1}{\beta^2}\left(\frac{\partial\beta}{\partial T}\right)_p\right\}_r-p_r p'\beta_r T_r\left\{\mathrm{K}+\frac{1}{\beta}\left(\frac{\partial\mathrm{K}}{\partial T}\right)_p\right\}_r\right.$$
$$\left.\left.+T'\beta_r^2T_r^2\left\{1+\frac{1}{\beta^2}\left(\frac{\partial\beta}{\partial T}\right)_p\right\}_r+O\left(\frac{\Delta T}{T_r}\right)^2\right]\right\}$$

And $$\frac{\frac{\rho_e}{\rho}-1}{\beta_r(\Delta T)_r}\simeq\frac{\Delta T}{(\Delta T)_r}\theta\left[1+\frac{\Delta T}{T_r}\theta\frac{\beta_r T_r}{2}\left\{1+\frac{1}{\beta^2}\left(\frac{\partial\beta}{\partial T}\right)_p\right\}_r+O\left(\frac{\Delta T}{T_r}\right)^2\right.$$
$$\left.+T'\left\{\left(\frac{T}{\beta}\left(\frac{\partial\beta}{\partial T}\right)_p\right)_r+O\left(\frac{\Delta T}{T_r}\right)^2\right\}-p'\left\{\left(\frac{p}{\beta}\left(\frac{\partial\mathrm{K}}{\partial T}\right)_p\right)_r+O\left(\frac{\Delta T}{T_r}\right)\right\}\right] \tag{31}$$

Since for a thermally perfect gas $\beta=\frac{1}{T}$ where $\beta=\beta(T,p)$

$$\therefore\left(\frac{\partial\beta}{\partial T}\right)_p=-\frac{1}{T^2}\ \text{ and }\ \left(\frac{\partial\mathrm{K}}{\partial T}\right)_p=0 \tag{32}$$

We expand the later to first order in small quantities and obtain; to zero order in T' and p',

$$\frac{T}{\Delta T}\frac{\rho_e}{\rho}\frac{\beta}{\beta_r}\simeq\frac{T_r}{\Delta T}+\theta\left\{1+\beta_r T_r\left(1+\frac{1}{\beta^2}\left(\frac{\partial\beta}{\partial T}\right)_p\right)_r\right\}+O\left(\frac{\Delta T}{T_r}\right) \tag{33}$$

Using (3-33) in equation (3-34), we have

$$\frac{T}{\Delta T}\frac{\rho_e}{\rho}\frac{\beta}{\beta_r}\simeq\frac{T_r}{\Delta T}+\theta\left\{1+\beta_r T_r\left(1-\frac{1}{\beta^2T^2}\right)_r\right\}+O\left(\frac{\Delta T}{T_r}\right),\ \frac{T_r}{\Delta T}+\theta=\frac{T_r}{\Delta T}+\theta\frac{T_r}{T_e} \tag{34}$$

[Since by zero order in $\frac{X+CY}{L}$, $\frac{T_r}{T_e}=1$].

Two straightforward examples of representative exterior property fluctuations are examined here for differences in surface temperature and external circumstances. These situations have effects that are similar to those of stress work. These are the circumstances where temperature and entropy are constants in the first place. For each of these two scenarios, ideal gases both thermally and calorically and general fluids undergoing slight states changes are taken into consideration.

For convenience in the following analysis, we write $\frac{X+CY}{L} = x + Cy$]

Expansions for $x+cy<<1$, three cases of exterior properties and ΔT variations are considered for perfect gases and general fluids. These are

(1) $T_e = T_r$ = constant, T_w = constant so that $\Delta T = (\Delta T)_r$ = constant

(2) S_e = constant, $\Delta T = (\Delta T)_r$ = constant .

(3) S_e = constant, T_w = constant so that $\Delta T = (\Delta T)_r - T_r\left[\left(\frac{T_e}{T_r}\right) - 1\right]$ = constant.

Since we have chosen for simplicity to consider exterior properties and ΔT variations which are governed by the length scale L. The u-momentum, v-momentum and energy equations can be written entirely in terms of the independent variables (Φ, x, y), the latter two variables being generally small compared with unity. Here $x + Cy = \frac{X+CY}{L}$

which implies that $x = \frac{X}{L}$ and $y = \frac{Y}{L}$, but $x+Cy<<1$ considering $0<C<1$,

so that $x = \frac{X}{L}$ is less than 1 and also $y = \frac{Y}{L}$ is less than 1 .

Now we have to expand the dependent variables F, S and θ by straight forward expansion in integer powers of $(x+Cy)$.

Thus, the assumed expansions are,

$$F(\Phi; x+Cy) = F_0(\Phi) + (x+Cy)F_1(\Phi) + O(x+Cy)^2$$

$$S(\Phi; x+Cy) = S_0(\Phi) + (x+Cy)S_1(\Phi) + O(x+Cy)^2, \theta(\Phi; x+Cy) = \theta_0(\Phi) + (x+Cy)\theta_1(\Phi) + O(x+Cy)$$

Expressions for the additional dependent variables contained in (20) to (22) which are found by the following way,

$$\begin{cases} \dfrac{\frac{\rho_e}{\rho} - 1}{\beta_r(\Delta T)_r} = \theta_0 + (x+Cy)\left(\theta_1 + A(\Phi)\right) + O(x+Cy)^2, \\ \dfrac{\rho\mu}{\rho_r\mu_r} = B(\Phi)\{1 - (x+Cy)D(\Phi)\} + O(x+Cy)^2 \\ \dfrac{1}{\Delta T}\left(\dfrac{C_{p_r}}{C_p}\dfrac{\rho_e}{\rho}\dfrac{T\beta}{\beta_r}\dfrac{g_\xi}{G} + \dfrac{\partial T_e}{\partial x} + \theta\dfrac{\partial \Delta T}{\partial x}\right) = \theta_0 + E(\Phi) + O(x+Cy), \\ \dfrac{1}{\Delta T}\left(\dfrac{C_{p_r}}{C_p}\dfrac{\rho_e}{\rho}\dfrac{T\beta}{\beta_r}\dfrac{g_\eta}{G} + \dfrac{\partial T_e}{\partial y} + \theta\dfrac{\partial \Delta T}{\partial y}\right) = \theta_0 + H(\Phi) + O(x+Cy) \end{cases}$$

Also, $\frac{(\Delta T)_r}{(\Delta T)} = 1 + O(x+Cy)$ and $B(\Phi) = \left[1 + \left\{\left(\frac{\Delta T}{T}\right)\right\}\theta_0\right]^{(\omega-1)}$.

Here $P_{r_r} = P_r$, $P_r = P$, and $C_{p_r} = C_p$ are considered.

$\frac{\rho_r}{\rho} = \left[1 + \left(\frac{\Delta T}{T_r}\right)\Theta\right]$, Where $\Theta = 1 + \in\theta$

We calculate the values of A, D, H and E for three cases by similar way. These values are shown in the following table.

Table 1: Values of A, D, H and E.

	Case-i	**Case-ii**	**Case-iii**
A	0	θ_0	$\theta_0\left\{1+\frac{T_r}{(\Delta T)_r}\right\}$

D	$\frac{\gamma}{\gamma-1}-\frac{(\omega-1)\frac{(\Delta T)_r}{T_r}\theta_1}{1+(\omega-1)\frac{(\Delta T)_r}{T_r}\theta_0}$	$\frac{\gamma}{\gamma-1}+(\omega-1)-\frac{(\omega-1)\frac{(\Delta T)_r}{T_r}\theta_1}{1+(\omega-1)\frac{(\Delta T)_r}{T_r}\theta_0}$	$\frac{\gamma}{\gamma-1}+(\omega-1)-\frac{(\omega-1)\left\{\frac{(\Delta T)_r}{T_r}\theta_1+\theta_0\right\}}{1+(\omega-1)\frac{(\Delta T)_r}{T_r}\theta_0}$
E	$\frac{T_r}{(\Delta T)_r}\frac{g_\xi}{G}+\theta_0\left(\frac{g_\xi}{G}-1\right)$	$\left\{\frac{T_r}{(\Delta T)_r}+\theta_0\right\}\left(\frac{g_\xi}{G}-1\right)$	$\left\{\frac{T_r}{(\Delta T)_r}+\theta_0\right\}\left(\frac{g_\xi}{G}-1\right)$
H	$\frac{T_r}{(\Delta T)_r}\frac{g_\eta}{G}+\theta_0\left(\frac{g_\eta}{G}-1\right)$	$\frac{T_r}{(\Delta T)_r}\left(\frac{g_\eta}{G}-C\right)+\theta_0\left(\frac{g_\eta}{G}-1\right)$	$\frac{T_r}{(\Delta T)_r}\left(\frac{g_\eta}{G}-C\right)+\theta_0\left(\frac{g_\eta}{G}-1\right)$

For isothermal cases, in momentum equations and energy equations, we considered $B(\Phi)=1$ and $A(\Phi)=D(\Phi)=0.$ With the approximation $\omega=1$, it is implies that $B(\Phi)=1$, but from table we see that *A* and *D* is not equal to zero. It will be noted that fluctuations in the exterior and surface conditions give rise to the finite values of the variables A and D; Gebhart, (1962) ignores these effects. We can see from the talks in the preceding section that it is impossible to select a set of outside conditions that would cause both A and D to be zero at the same time. The energy equations' pressure work effects. $4(\theta_0+E(\Phi))F_0'\left(1+\frac{\Delta T}{T_r}(1+\in\theta_w)\right)$ and $4(\theta_0+H(\Phi))S_0'\left(1+\frac{\Delta T}{T_r}(1+\in\theta_w)\right)$ whose modified forms are neglected in Gebhart, (1962) analysis. It is seen from take 1 that, in most cases, the values of *A*, *D*, *E* and *H* are not small compared with viscous work effects.

Inherent in both the analysis of Gebhart, (1962) and that of Gebhart and Mollendrof, (1969) is the assumption that $\frac{(\Delta T)_r}{T_r}<<1.$ For convenience, we write $\in=\frac{(\Delta T)_r}{T_r}$ and also

$$\left(1+\frac{\Delta T}{T_r}(1+\in\theta_w)\right)\simeq\left(1+\frac{(\Delta T)_r}{T_r}(1+\in\theta_w)\right)=\left(1+\in(1+\in\theta_w)\right) \tag{36}$$

Substituting (36) in momentum equations and energy equations and then equate the co-efficient of $\in^0$ from both sides. By using the above expressions (36) and also applying the three cases in the *u*-momentum, *v*-momentum and energy equation and then also equating, we have the zeroth $(x+Cy)^0$ and first order $(x+CY)$ equation the following form

$$\left.\begin{aligned}&\left[\frac{BF_0''}{P_r}\right]'-2F_0'^2-2CF_0'S_0'+3F_0F_0''+3CS_0F_0''+4\theta_0\sin\delta=0\\&\left[\frac{BS_0''}{P_r}\right]'-2CS_0'^2-2F_0'S_0'+3F_0S_0''+3CS_0S_0''+4\theta_0\cos\delta=0\\&\text{and}\quad\left[\frac{B\theta_0'}{P_r}\right]'+3\left\{F_0+CS_0\right\}\theta_0'=0\end{aligned}\right\} \tag{37}$$

Boundary conditions are given below

$$F_0(0)=F_0'(0)=F_0'(\infty)=0,\ S_0(0)=S_0'(0)=S_0'(\infty)=0\ \text{and}\ \ \theta_0(0)=1,\ \theta_0(\infty)=0. \tag{38}$$

Similarly, the first order of x and y,

$$\left.\begin{array}{l}[B\{F_1'' - DF_0''\}]' - 8F_0'F_1' + 7F_0''F_1 - 6CS_0'F_1' + 7CS_1F_0'' \\ + 3(F_0 + CS_0)F_1'' - 2CF_0'S_1' + 4\{\theta_1 + A\}\sin\delta = 0 \\ \\ [B\{S_1'' - DS_0''\}]' - 8CS_0'S_1' + C(3 + 4C)S_0''S_1 - 6F_0'S_1' + 7F_1S_0'' \\ + 3(F_0 + CS_0)S_1'' - 2S_0'F_1' + 4\{\theta_1 + A\}\cos\delta = 0 \\ \\ \left[\dfrac{B\{\theta_1' - \theta_0'\}}{P_r}\right]' + 7(F_1 + CS_1)\theta_0' + 3(F_0 + CS_0)\theta_1' - 4(F_0' + CS_0')\theta_1 \\ -4(\theta_0 + E)F_0^1 - 4(\theta_0 + H)S_0' + B\{F_0''^2 + S_0''^2\} = 0\end{array}\right\} \quad (39)$$

The boundary conditions for the above first order *u*-momentum, *v*-momentum and energy equations are shown below.

$$F_1(0) = F_1'(0) = F_1'(\infty) = 0,\ S_1(0) = S_1'(0) = S_1'(\infty) = 0$$

$$\text{And}\quad \theta_1(0) = \theta_1(\infty) = 0 \quad (40)$$

The transformed equations can be solved with the help of the controlling parameters P_r δ , *A, B, C, D, E, H,* $(\frac{\Delta T}{T})_r$.

RESULTS AND DISCUSSION:

The non-linear and connected two-point boundary value issues are represented by a single set of equations and their corresponding boundary conditions, that is, one set of equations (37) with boundary conditions (38). Analytically solving them is challenging. Therefore, we use a process to obtain the numerical solutions. We solve the equations numerically by using the Runge-Kutta Merson method in conjunction with the Runge-Kutta shooting method and the Swigert iteration technique. By utilizing the straightforward expansion, the higher order non-linear differential equations are essentially reduced to a series of first order initial value problems with corresponding boundary conditions. To solve the zero order equations (37) with their typical boundary conditions, the Runge-Kutta shooting method, the Runge-Kutta-Merson method, and the Swigert iteration technique (i.e., guessing the missing values) are taken into consideration. The nature of the flow processes resulting from free convection flow will be covered in the next section for a range of controlling parameter values (Pr, *C*, *B*, $(\Delta T/T)_r$). The velocity and temperature profiles for the zero order transformed similarity equations are shown in **Fig. 1** and **2**. The impacts of skin friction factors and heat transfer coefficient for the zero order with a rise in *C* are displayed in **Fig. 5** and **6**. The velocity components rise as *C* increases and vice versa, as seen in the given **Fig. 2a** and **2b**. However, what's interesting in this case is that the two velocity distribution components don't change in the same way. Temperature profiles are seen to exhibit minor behavior for the *C*-variations in **Fig. 2c**. This suggests that the additive characteristic parameter (length), *C*, is not the only factor influencing the rate of heat transfer. Comparable outcomes (shown in **Fig. 3a** and **3b**) are discovered to be relevant for the water scenario as well (Pr = 7.0). The fixed values of controlling parameters δ and *B* are 15^0 and 1 respectively. In the case of temperature profiles (**Fig. 3c**) the additive characteristic parameter (length) *C* shows its remarkable change for water (Pr = 7.0) than air (Pr = 0.72). Temperature decreases gradually with the increasing values of *C* when another controlling parameter (δ = 15^0 and *B*=1.0) remain fixed. **Fig. 4a** and **4b** displayed the dimensionless velocity distributions along u,v-directions for several values of C also the fixed values of controlling parameters. The physical meaning of **Fig. 4a**, **4b** are same as **2a**, **2b** and if **Fig. 4c** is compared with **2c** we have discerned the same rule. **Fig. 6a** and **6b** respectively show the fluctuation of skin friction $\left(F''_0(0),\ S''_0(0)\right)$ and the heat transfer coefficient $\left(-\theta_0'(0)\right)$ for the additive characteristic length parameter C and Pr=7.0. Once more **Fig.7a** and **7b** respectively show the fluctuation of skin friction $\left(F''_0(0),\ S''_0(0)\right)$, and **Fig. 7c** shows the heat transfer coefficient $\left(-\theta_0'(0)\right)$ for the change of the additive characteristic length parameter C and Pr=0.72.

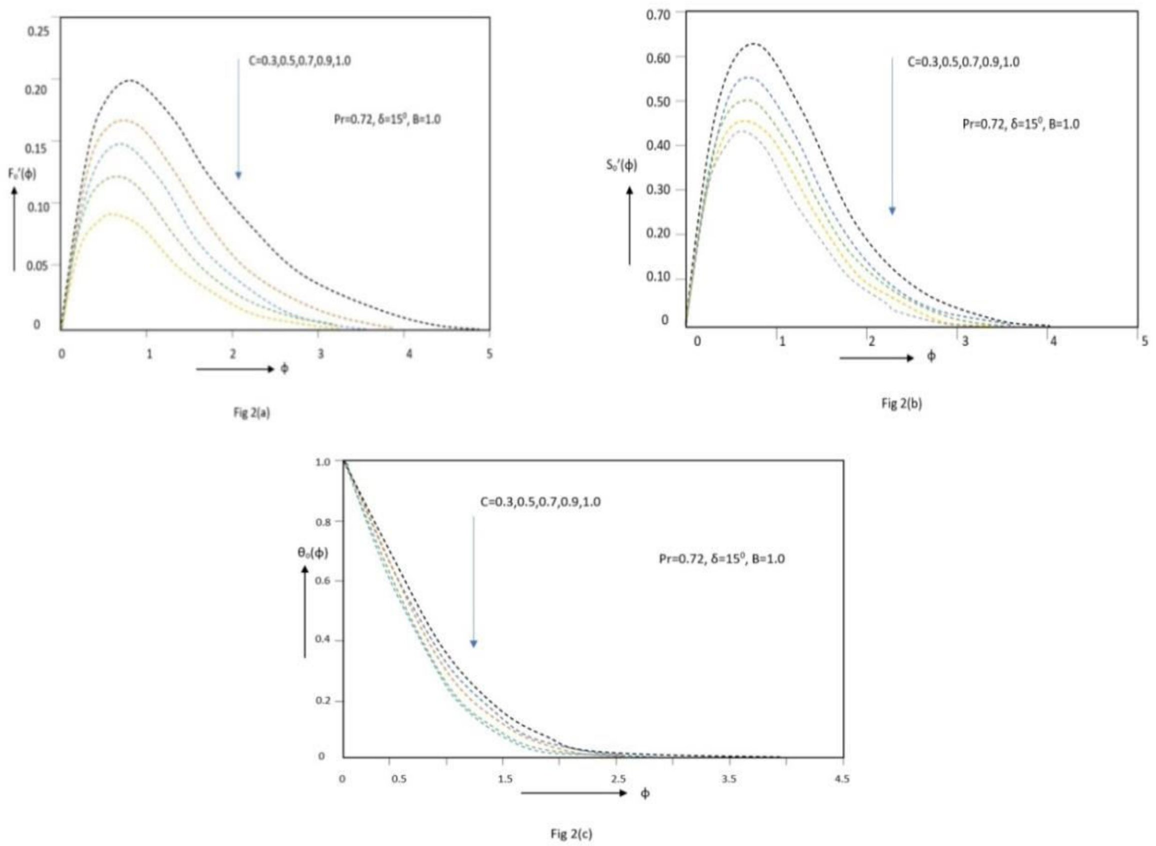

Fig. 2a, **Fig. 2b** and **Fig. 2c:** are the dimensionless u, v-velocity distribution and dimensionless temperature distributions for several values of $\boldsymbol{C}$ = (0.3, 0.5, 0.7, 0.9, 1.0) and the fixed values of Pr = 0.72, $\delta = 15°$ and B=1.0; for the equation of zero order (37).

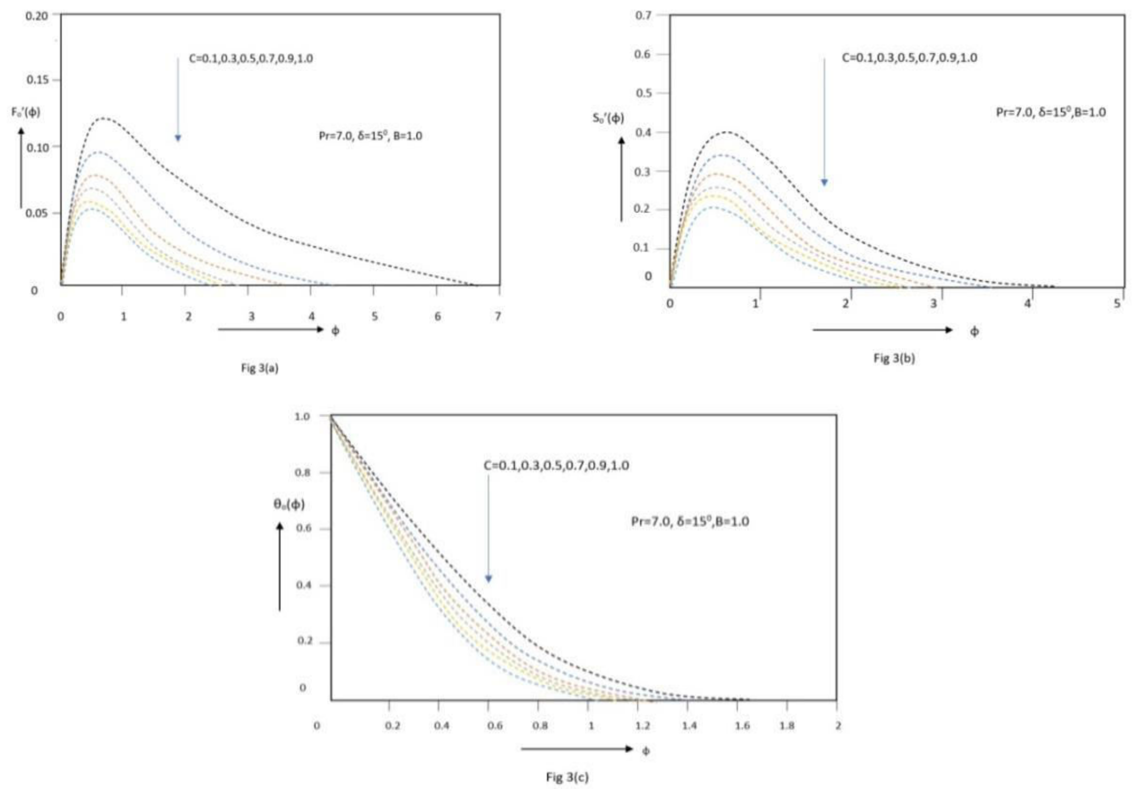

Fig. 3a, **Fig. 3b** and **Fig. 3c:** are the dimensionless u-velocity, v-velocity distribution and dimensionless temperature distributions for several values of $\boldsymbol{C}$ = (0.1,0.3,0.5,0.7,0.9,1.0) and the fixed values of Pr = 7.0, $\delta = 15°$ and B=1.0; for the equation of zero order (37).

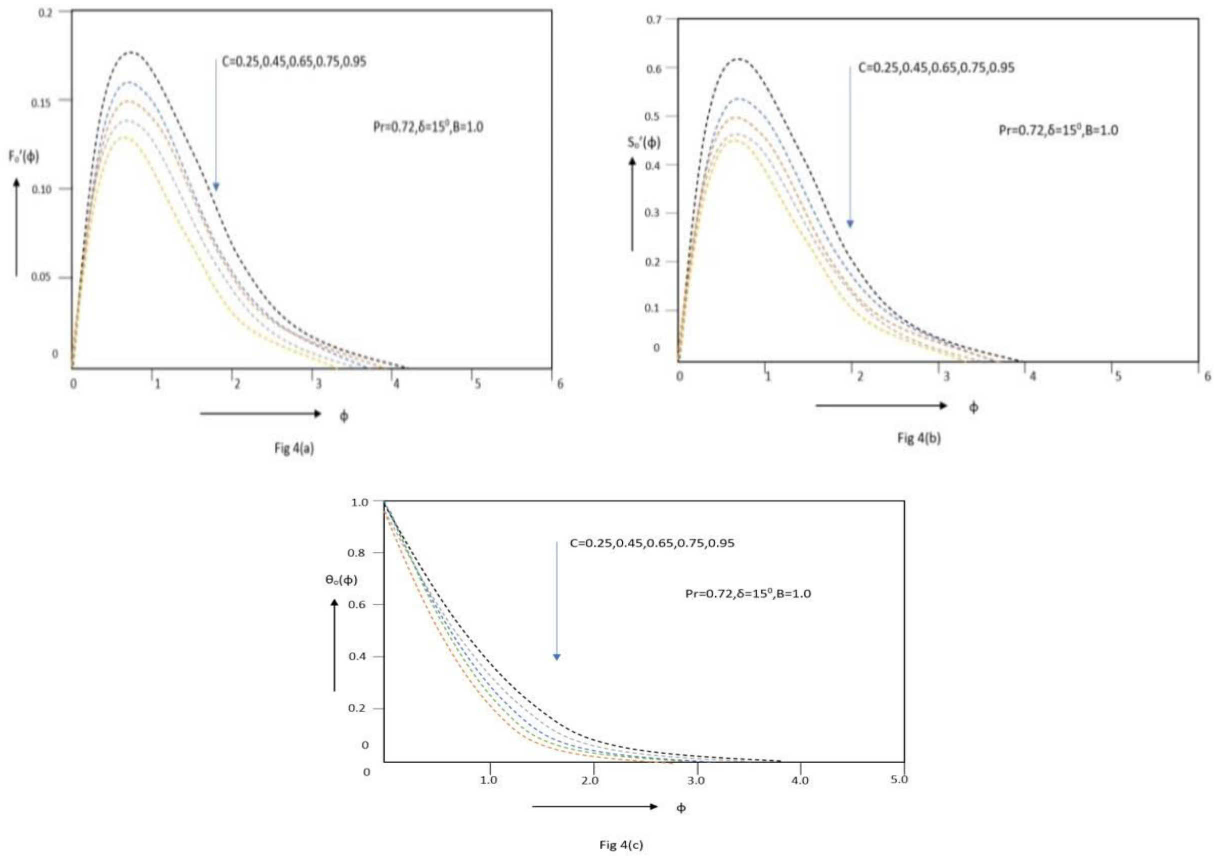

Fig. 4a, **Fig. 4b** and **Fig. 4c**: are the dimensionless u-velocity, v-velocity distribution and dimensionless temperature distributions for several values of $\boldsymbol{C}$ = (0.25, 0.45, 0.65, 0.75, 0.95) and the fixed values of Pr = 0.72, $\delta = 15°$ and B=1.0; for the equation of zero order (37).

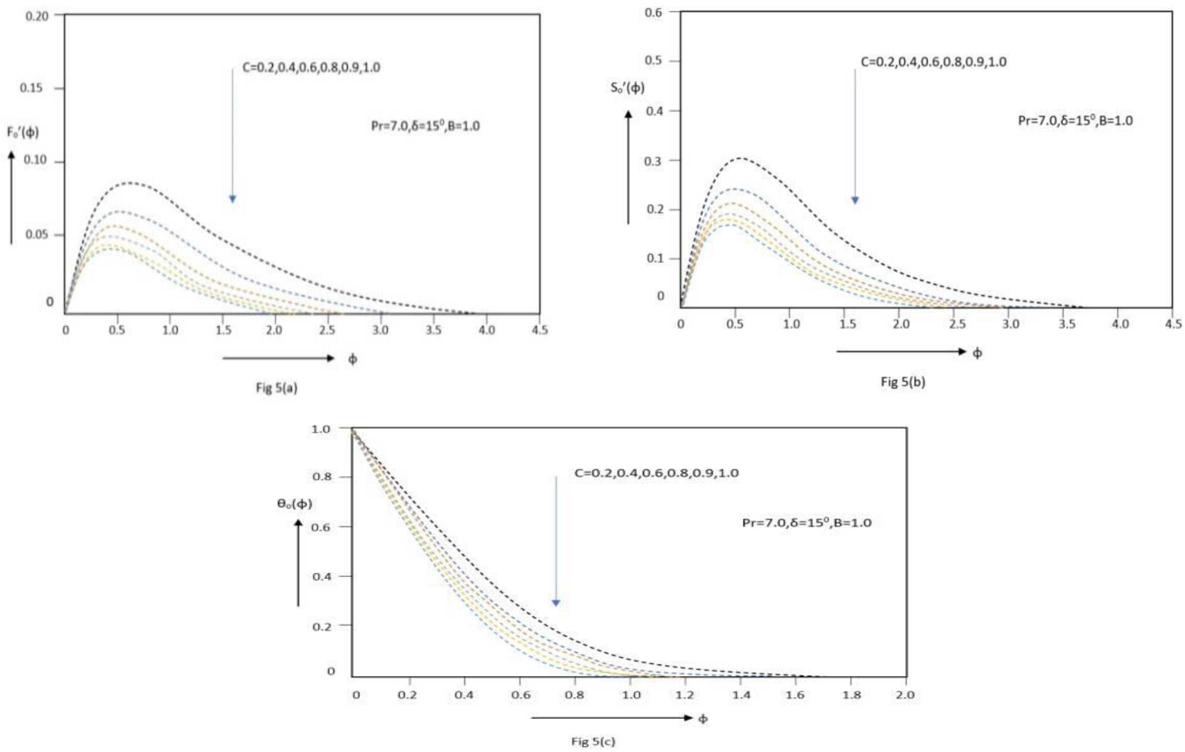

Fig. 5a, **Fig. 5b** and **Fig. 5c:** are the dimensionless u**-**velocity, v-velocity distribution and dimensionless temperature distributions for several values of C = (0.2, 0.4, 0.6, 0.8, 0.9, 1.0) and the fixed values of Pr =7.0, $\boldsymbol{\delta = 15}$°and B=1.0; for the equation of zero order (37).

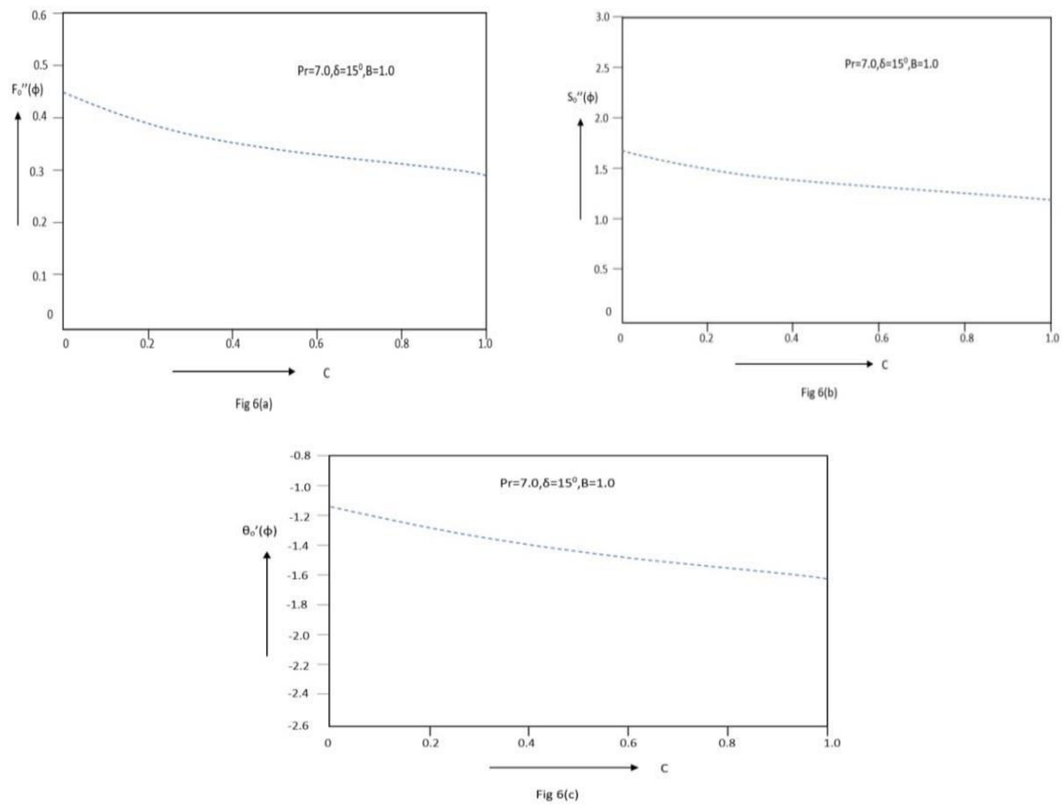

Fig. 6a, **Fig. 6b** and **Fig. 6c:** are the skin friction factors along u-direction, v-direction and Heat transfer coefficient against C = (0.0 to 1.0), Prandlt's number Pr =7.0, B=1.0 for the equation zero order (37).

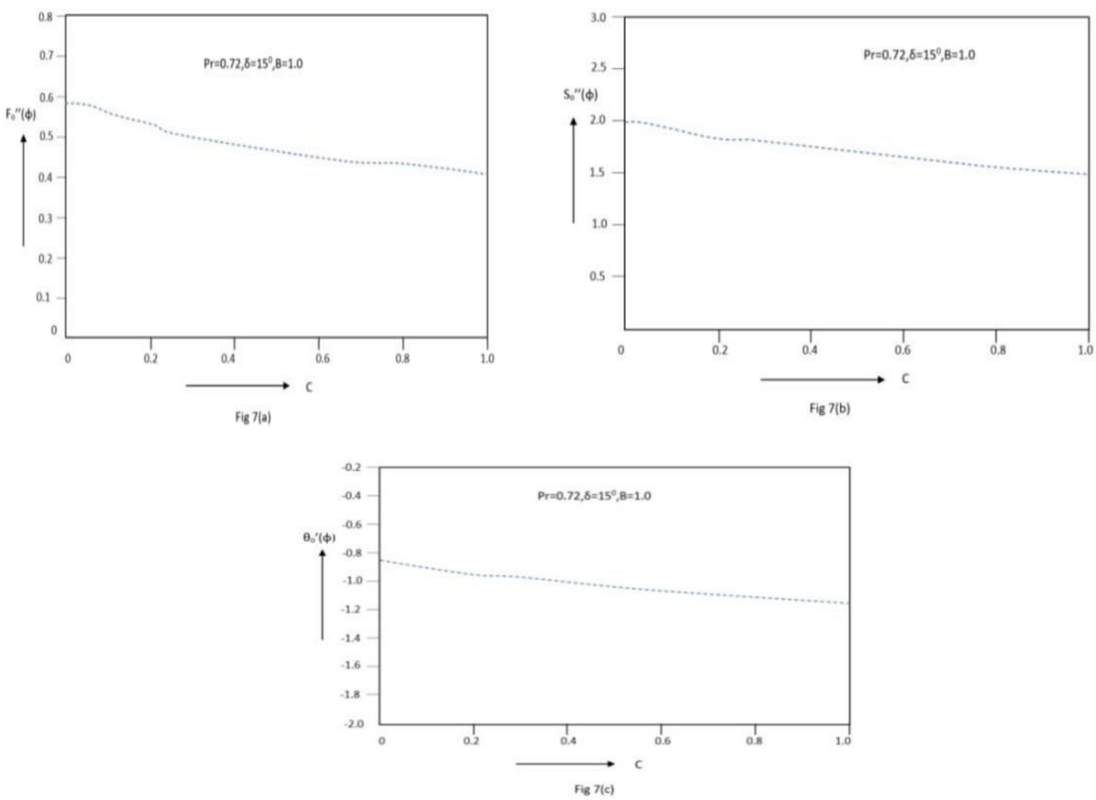

Fig.7a, **Fig.7b** and **Fig.7c**: are the skin friction factors along u-direction, v-direction and Heat transfer coefficient against C = (0.0 to 1.0), Prandlt's number Pr = 0.72, B=1.0 for the equation zero order (37)

Table 2: The influence of skin friction factors and heat transfer coefficient for Pr=7.0, *B*=1.0 and δ=15^0.

C	$F_0''(0)$	$S_0''(0)$	$-\theta_0'(0)$
0.0	0.46261	1.62559	1.08471
0.1	0.43051	1.52407	1.16531
0.2	0.40476	1.45042	1.23114
0.3	0.38474	1.38781	1.28912
0.4	0.36821	1.33662	1.34031
0.5	0.35442	1.29315	1.38648
0.6	0.34274	1.25531	1.42863
0.7	0.33263	1.22188	1.46745
0.8	0.32390	1.19230	1.50350

Table 3: The influence of skin friction factors and heat transfer coefficient for Pr=0.72, B=1.0 and δ=15^0.

C	$F_0''(0)$	$S_0''(0)$	$-\theta_0'(0)$
0.0	0.56560	1.94367	0.78953
0.1	0.54109	1.85756	0.83939
0.2	0.51222	1.77096	0.88448
0.3	0.47852	1.72257	0.92012
0.4	0.45802	1.66199	0.95562
0.6	0.42697	1.56082	1.01790
0.7	0.41479	1.51922	1.04537
0.8	0.41104	1.45956	1.07113
0.9	0.40033	1.42685	1.09211

The influence of skin friction factors and heat transfer coefficient is shown in numerical **Table 2** with variations in *C* and Pr = 7.0, δ = 15^0, *B* = 1.0, and other fixed regulating parameters. The impacts of skin friction factors and heat transfer coefficient with adjustment of the additive characteristic length parameter *C* and other fixed regulating parameters are also shown for air in numerical **Table 3**.

CONCLUSION:

The square side flat surface has its diagonal vertical for δ = 45^0 and *C* = 1.0. Here, in this arrangement of natural flow, F and S become identical, and the u- and v-momentum equations likewise coincide. The v-velocity component is more vertical than the u-velocity component here, when δ=15^0, i.e., one of the y-direction edges forms an angle δ with the vertical. Takes a higher value than for the same values of *C* because of this. The results of the current investigation are consistent with those of Ackroyd, (1974) for zero order similarity solutions with δ = 90^0 and *C* = 0. The identical result as Gebhart, (1962) is obtained once more when B=1.0, *C* = 0, and δ = 90^0. We find that, under the influence of the additive characteristic length *C* and acute angle δ=15^0, the skin friction factors and heat transfer coefficient for zeroth order differ slightly from those of Ackroyd, (1974) and Gebhart, (1962).

ACKNOWLEDGEMENT:

The authors are grateful to the Department of Mathematics, DUET, Gazipur, Bangladesh.

CONFLICTS OF INTEREST:

There is no conflict of interest between the authors.

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