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Theoretical Results of the Extended Gamma Function and Its Applications

Bachioua Lahcene[1]*, Al Mahmud Al Mamun[2], Md. Rasel Hossain[3], and Soriful Islam[4]

[1]Dept. of Mathematics, Faculty of Exact Sciences, University of Bejaia, Algeria; [2]Dept. of Computer Science and Engineering, Prime University, Bangladesh; [3]Dept. of Statistics, Noakhali Science and Technology University, Noakhali, Bangladesh; and [3]Dept. of Applied Mathematics, Gono Bishwabidyalay, Savar, Dhaka 1344, Bangladesh.

*Correspondence: drbachioua@gmail.com (Dr. Bachioua Lahcene, Associate Professor, Dept. of Mathematics, Faculty of Exact Sciences, University of Bejaia, Algeria).

ABSTRACT

In recent years, there has been a lot of interest in the special functions of extended functions and their uses, some of which define the totality of partial analyses, provide useful tools for describing natural phenomena, and are thus more suitable for describing some applicable models. This work illustrates some of the rich theoretical and applied behaviors found in models of special functions, especially expansion-generalized gamma delta, and approaches to generalizing integrals and derivatives more comprehensively, through the weights provided by extended gamma functions. The researcher tried to link all the basic modifications that were obtained previously, and with a summary of the modifications that appeared on the most important special functions related to the extended generalized gamma function and the special functions overlapping with it related to the fractional calculus and more results about the generalized gamma function that occur in the diffraction theory, and some special functions related to fractional functions. Calculus and more results about the extended gamma function that occurs in diffraction theory that occurs in diffraction theory in most applications with full control over diffraction access to functions (of different scale) diffraction of light waves, in traditional diffraction theory.

Keywords: Extended, Gamma functions, Transformation functions, Fractional calculus, and Control theory.

INTRODUCTION:

The followers of the gamma function will find its historical roots back in an ancient time, in the year 1727, when Leonard Euler was able to study the function for the first time (Davis, 1959). In the year 1730, Christian Goldbach was able to modify the period of defining the integral of the gamma function for the period [0, 1], where he reformulated it in an equivalent form to the first form using the modification of the gamma function (Sandifer, 2007).

During the eighteenth century, the derivation of many properties and applications of the gamma function appeared in various engineering, physical and financial sciences. One of the most important special functions has applications in many fields of science, for example, analytic number theory, statistics, and physics. We recall that the gamma function is defined in the simple form (Yousif and Arbab, 2022; Larry, 1992).

$$\Gamma(x) = \int_0^{+\propto} t^{x-1}\, e^{-t} dt\,.\ Where\ Re(x) > 0$$

Using the modulation in the abbreviated domain [0, 1], we get the formula

$$\Gamma(x) = \int_0^{+1} [-\log(t)]^{x-1}\, dt\,.\ Where\ Re(x) > 0$$

It can be easily observed that there is relationships linking the definition of the gamma function with the integral formula of the Laplace transform, as -

$$L(f)(x) = \int_0^{+\infty} f(t)\, e^{-xt} dt$$

Using a simple substitution when $Re(x) > 0$, we find that the gamma function is a special case, as follows:

$$\Gamma(x) = L(f)(1) = \int_0^{+\infty} f_x(t)\, e^{-t} dt.\ Where\ f_x(t) = t^{x-1}$$

Until the mid-20th century, mathematicians relied on hand-made tables; in the case of the gamma function, notably a Table computed by Gauss in 1813 and one expressing its mathematical formula by Legendre in 1825 (Wikipedia, 2015). Use respectively the change of variables for $Re(x) > 0$ we get the function equation in the new form:

$$\Gamma(x) = \int_0^{+\infty} t^{x-1}\, e^{-t} dt = 2\int_0^{+\infty} t^{2x-1}\, e^{-t^2} dt$$

As a reminder, we note that the researchers, both Euler and Legendre, studied the beta function and defined it by the following integral form:

$$B(x,y) = \int_0^{+1} t^{x-1}(1-t)^{y-1}\, dt = \frac{\Gamma(x)\Gamma(y)}{\Gamma(x+y)}.\ Where\ Re(x) > 0.Re(y) > 0$$

In continuation to the basic and wonderful results shown by previous studies on the gamma function referred to earlier in various fields, especially in many theoretical and practical applications, many researchers are encouraged to suggest generalized formulas for this function. Abdul Moiz & Bachioua, (2017) and Kobayashi, (1991) proposed an additional generalized formula for the gamma function in terms of three basic parameters, & it was expressed by the following formula:

$$\Gamma_r(x,n) = \int_0^{+\infty} \frac{t^{x-1}}{(t+n)^r} e^{-t} dt\ .\ Where\ Re(x) > 0.r > 0.Re(n) > 0$$

It is obvious that this function reduces to a simple case $\Gamma(x) = \Gamma_0(x,n)$ of gamma function. This generalized formula has achieved useful functions in the problems of diffraction theory, and the problems of corrosion and rust of metal production machines, and for this, the formula of the generalized function was modified by Kobayashi, (1991) and its formula became with four parameters as follows:

$$\Gamma_r(x,n,\lambda) = \int_0^{+\infty} t^{x-1}\, (t+n)^{-r} e^{-\lambda t} dt\ .Where\ Re(x) > 0.r > 0.\lambda \geq 0.Re(n) > 0$$

The researchers allowed the modified formula in terms of four parameters obtained to find the basic constraints that allow neutralizing some side aspects of the formula of the integrative function, as it appears the possibility of excluding the exponential part of the integral, which is represented in this case by the following formula:

$$\Gamma_r(x,n,0) = \int_0^{+\infty} t^{x-1}\, (t+n)^{-r} dt$$

The researchers Chaudhry & Zubair, (2001) studied in depth the proposed new case for the new formula of the generalized gamma function in terms of the two parameters, which was formulated in terms of the integral function as follows in the formula:

$$\Gamma_r(x) = \int_0^{+\infty} t^{x-1}\, e^{-t-\frac{r}{t}} dt;\ whereRe(x) > 0$$

Recently, Bachioua studied with interest and in-depth the proposed cases and extended the formula for the gamma function referred to earlier. In 2004, he proposed an expanded extension of six coefficients of the gamma function, where he defined the basic conditions for the convergence of the proposed extended integral (Bachioua, 2004), which he presented with the following integral formula:

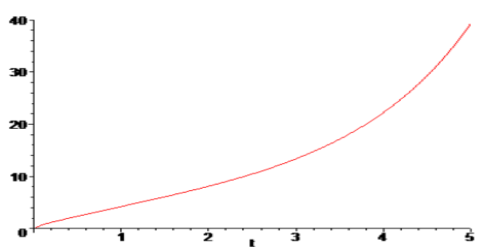
$$\Lambda_r(\alpha,m,n,p,k,\lambda) = \int_0^{+\infty} t^{\alpha-1}\, (t^m+n)^{-r} e^{-\lambda t^{p-1}} dt\ ,\quad Where\ Re(\alpha), Re(n) > 0, r > 0, \lambda, p \geq 0$$

Through the general formula and by fixing the values of the parameters, it is noted that it is simply possible to obtain most of the previous formulas with transformations of the values of the six parameters included in the proposed function model. Expanding the scope of including the expected formulas, the exponential part of the proposed formula was expanded and modified, as the researcher presented a new modified formula in the proposed model with the following formula:

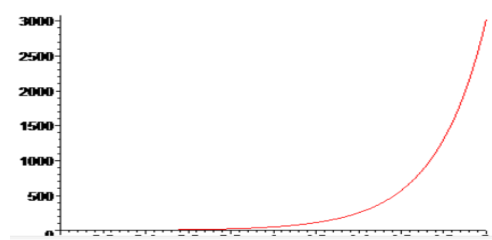
$$\Lambda_r(\alpha,m,n,p,k,\lambda,\gamma) = \int_0^{+\infty} t^{\alpha-1}\, (t^m+n)^{-r} e^{-\lambda t^{p-1}+\gamma t^{-k}} dt$$

The new formulations of the gamma function have confirmed its importance in many applications, especially in probability distributions, stochastic models, entropy, and other special physical and integral applications (Andrews, 1985; Lebedev, 1972). It is recommended to specify the estimations of the values of the basic parameters through the finite domain, which makes the integral convergent to the expanded integral. The researcher indicates that by reading the references related to the topic of the gamma function, he found that there are many important studies that dealt with the subject and the applications of the extended and regular gamma function, and it is found in many references that dealt with this function in various simple topics of life. The results obtained for the applications of the gamma function prove the tremendous success of the gamma function. Several authors have identified and discussed different types of gamma functions in recent years. More recently, Kobayashi considered flat wave diffraction by tape using the Wiener-Hopf technique (Kobayashi, 1991; Abramowitz, 1972). The generalized form of the hyper geometric function has been investigated by Malovi chko, (1976).The gamma function by inserting a hyper geometric Gaussian function (Saboor and Ahmad, 2012) into it. Agarwal and Kalla, (1996) define and study the generalized gamma distribution. They used a modified form of Kobayashi's. Al-Musallam and Kalla, (1997 and 1998)) extended the gamma function by engaging a supra-geometric Gaussian function. Sexena and Kalla, (1960) discuss a hyper-geometric gamma function Saxena & Kalla, (2001). Although the Wiener-Hopf technique (Noble, 1988) is a powerful tool for studying wave scattering and diffraction problems related to strips and slits (Kobayashi, 1991). We recall well that the field of use of the generalized gamma function is a wide range, as we mention the fractional calculus, diffraction, control theory, tracking physical waves, and tracking entropy calculations (Bachioua, 2006 and 2021)), and the research is still ongoing to test this function in integrative approximations and study the probability distributions associated with phenomena Randomness that falls under the field of generalized probability distributions, and many researchers have referred to some of these applications and are referred to in the references (Caputo, 1967; Guseinova and Mamedov, 2007).

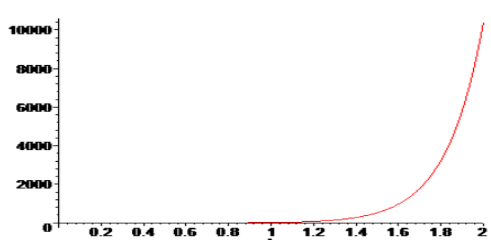
Saboor *et al.* (2012) identified the bivariate gamma function and its corresponding density function and discussed some of its properties. Chaudhry, Aslam, & Zubair, (2001) provided an extended Gamma function. It is interesting to note that gamma, generalized gamma, Gamma-type functions, and extensions identified by (Ghitany *et al.,* 2012; Zamel *et al.,*2012; Saboor *et al.,* 2012; Ali *et al.,* 2001; Jade, 1953), Recently A. Al-Zamel has introduced a new gamma type (Al-Zamel, 2001 & Bachioua and Abdulmoiz, 2017) follow as special cases of our standard form of Extended Gamma Functions. **Fig.** (**a:1**), (**a:2**), (**a:3**), (**a:4**), (**a:5**), (**a:6**), (**a:7**) (**a:8**), (**a:9**), (**a:10**) give a clear idea, which are drawn in the plan.



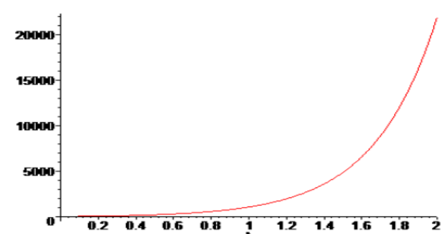
Graph (a;1): Extended Gamma function
$\alpha = 1/2; m = 2, n = 1, p = 2, k = 1/2, r = 1, \lambda = 1, \gamma = 2$



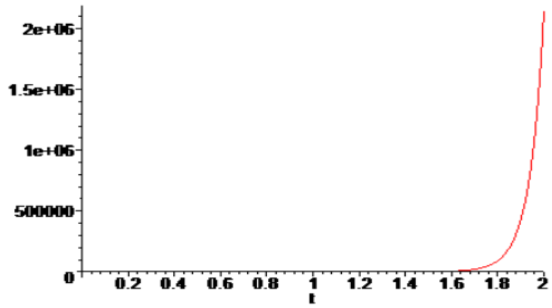
Graph(a;2): Extended Gamma function
$\alpha = 1/2; m = 2, n = 1, p = 2, k = 1/2, r = 1, \lambda = 5, \gamma = 1$



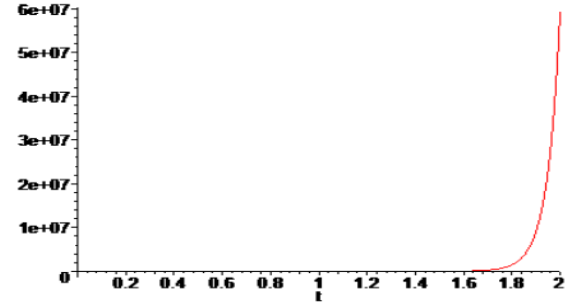
Graph(a;3): Extended Gamma function
$\alpha = 3; m = 2, n = 2, p = 2, k = 1/2, r = 1, \lambda = 5, \gamma = 1$



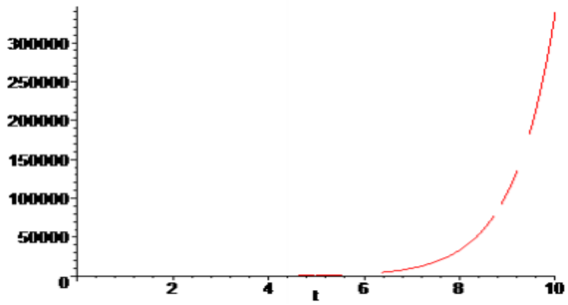
Graph(a;4): Extended Gamma function
$\alpha = 1; m = 0, n = 2, p = 2, k = 1, r = -1, \lambda = 3, \gamma = 4$



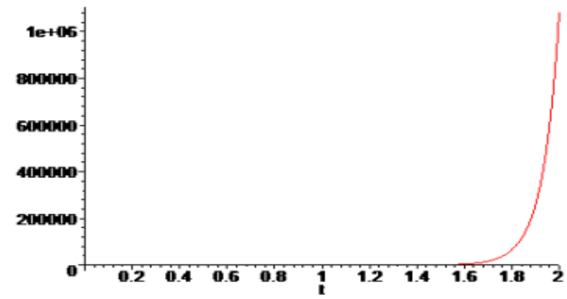
Graph(a;5): Extended Gamma function
$\alpha = 0; m = 2, n = 0, p = 3, k = -1, r = -1, \lambda = 3, \gamma = 1$



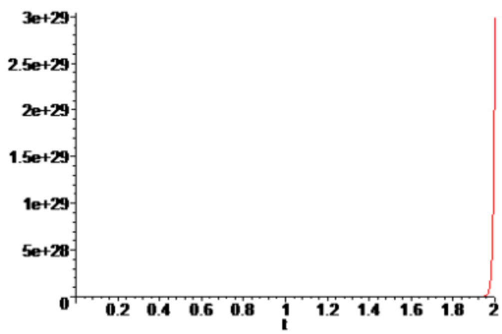
Graph(a;6): Extended Gamma function
$\alpha = 5; m = 1, n = 0, p = 3, k = -1, r = -3, \lambda = 3, \gamma = 1$



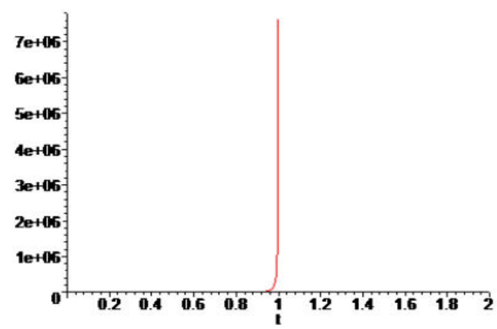
Graph(a;7): Extended Gamma function
$\alpha = 1/4; m = 2, n = -1, p = 2, k = -2, r = -1, \lambda = 1, \gamma = 0$



Graph(a;8): Extended Gamma function
$\alpha = 1/4; m = 2, n = -1, p = 2, k = -1, r = -1, \lambda = 2, \gamma = 3$



Graph(a;9): Extended Gamma function
$\alpha = 3; m = 2, n = -2, p = 2, k = -3, r = -2, \lambda = 3, \gamma = 4$



Graph(a;10): Extended Gamma function
$\alpha = 1; m = 1, n = -1, p = 1, k = -2, r = 2, \lambda = 4, \gamma = 5$

It is observed through the various graphs obtained to extend the generalized gamma function when fixing some values for the eight parameters that the extended gamma function is defined in terms of the eight basic parameters, and it is clear that the shape of this function extends on the horizontal surface of some parameters and takes the form of the function values represented in The vertical line, where the figure shows the clear exponential form with some inflections, and therefore it is necessary to expand the target special cases, and connect them to the area of convergence of integration defined by the approximation domain of the expanded gamma function with the new extended generalized gamma function.

Some Transform and Extended GAMMA Function

We remember well that the Laplace transform is among the algebraic effects expressed by the process of integration. The transformation is named after the French scientist Laplace, who lived in the nineteenth century. (Abramowitz and Stegun, 1972), where they are performed on mathematical functions in order to convert them from one specific field to another, usually the conversion from the time domain to the frequency domain, which is similar to the Fourier transform operator, except that it was developed independently of other transformations (Niazai *et al.*, 2022).

The Laplace transform is very useful in the analysis of systems extensively in linear functions, unlike the Fourier transform, which is commonly used in the analysis of signals and topics related to topics in electronics; in addition to that it is used to solve complex differential equations because it transforms them into algebraic equations:

$$L(f)(x) = \int_0^{+\propto} f(t)\, e^{-xt} dt$$

Laplace Transforms can return the extended general gamma function form:

$$\Lambda_r(\alpha, m, n, p, k, \lambda, \gamma) = \int_0^{+\propto} t^{\alpha-1}\,(t^m + n)^{-r} e^{-\lambda t^{p-1} + \gamma t^{-k}} dt$$

$$= \int_0^{+\propto} t^{\alpha-1}\,(t^m + n)^{-r} e^{-t\{\lambda t^{p-2} + \Upsilon t^{-k-1}\}}$$

$$= L(f)(\lambda t^{p-2} + \Upsilon t^{-k-1}),$$

$$Where\ f_x(t) = t^{\alpha-1}(t^m + n)^{-r}$$

There are many characteristics that characterize the Laplace transform, which are known to specialists to facilitate its use in the specified ranges, especially in the analysis of linear systems in the field of calculus. The most important of which are cases to facilitate the solution of differential equations and their associated fields, which can be expressed in the cases guaranteed by the transformation. We also mention that there are researchers who define the beta-Euler transformation as an extension of the private transformation, and this transformation is expressed in the following integral form:

$$\beta\{(f)(x);\ \alpha.r\} = \int_0^{+\propto} x^{\alpha-1}\,(1 - x)^{-(r-1)-1} f(z)dz, Re(\alpha), Re(r) > 0$$

We deny that the Laguerre transformation is generally formulated using polynomial functions, and is expressed by the following integral:

$$L\{(f)(x)\} = \int_0^{+\propto} e^{-x}\, x^{\mu} L_n^{\mu}(x) f(x) dx,$$

$$Where\ L_n^{\mu}(x)\,, n \geq 0$$

Fractional Calculus and Extended Generalized GAMMA Function

In this part we always adopt the fractional differentiation involved in the following Caputo type [22] defined by the following formula:

$$D^{\alpha}\theta(x) = J^{s-\alpha}\theta^{(s)}(x), \alpha \epsilon IR^+$$

The value of α is taken in the range [0.1], this parameter represents the fractional value that expresses the fractional order of differentiation. $\theta^{(s)}$ Represent's the s-order of the derivative, and $\theta(x)$ indicates the minimum integer not less than the fractional parameter, and J^{μ} is described by -

$$J^{\mu}\vartheta(x) = \big(\Gamma(\mu)\big)^{-1} \int_0^{x} (x - \chi)^{\mu-1}\, \vartheta(\chi) d\chi, \mu \epsilon IR^+$$

Where $\Gamma(x) = \int_0^{+\propto} t^{x-1}\, e^{-t} dt\,. Where\ Re(x) > 0$

We know very well that from the references reviewed, many generalizations and additions have been made to the generalized gamma function by extension, and that is why we ask here what changes and additions these extensions and additions can provide to the number of parameters when the gamma function is expressed in its generalized & extended form. We try to study here the first case proposed by Agarwal *et al.* (1996), they refer to the following formula (Agarwal & Kalla, 1996).

$$\Gamma_r(\mu, n, \alpha) = \int_0^{+\propto} t^{\mu-1}(t + n)^{-r}\, e^{-\alpha t} dt = \alpha^{r-\mu}\Gamma_r(\mu, \alpha n)$$

In an important simple special case, we find that $\Gamma_0(\mu)$=$\alpha^{-\mu}\Gamma(\mu)$,

Analytical Properties of Fractional Calculus

The idea of partial calculus dates back to the seventeenth century (Dugowson, 1994), and by the twenty-first century, many different partial calculus factors have been identified over the past years. Those interested in this topic note that many methods and mechanisms have been proposed that facilitate the process of defining fractional integrals, and the derivatives associated with them, as it is noted that some of them are equivalent to the definition of Riemann-Liouville, while others show slight differences in the formula referred to, which allows the search for factor formulas that include the different cases in a comprehensive formula. The integrals and derivatives of Riemann-Liouville are respectively known by the following mathematical formulas (Dugowson, 1994).

$${}^{RL}_{a}I_t^{v} f(t) \coloneqq \big(\Gamma(v)\big)^{-1} \int_a^{t} (t - u)^{v-1} f(u) du, \quad Re(v) > 0$$

$${}^{RL}_{a}D_t^{v} f(t) \coloneqq \frac{d^n}{dt^n}\, {}^{RL}_{a}I_t^{n-v} f(t), n \coloneqq \lfloor Re(v) \rfloor + 1\,, \quad Re(v) \geq 0$$

It is often expressed in the form of the Riemann-Liouville model and in the form of an infinite convergent series, which is convergent (Baleanu and Fernandez, 2018). In order to modify the form of modern terms & concepts in the subject & unify them in a general way, we now begin to review some of these basic definitions.

Definition (1)

We present the formula for defining the reduced partial integrals of the interval $[a.b]$, and the complex parameters α, β ∈ IC, providing that $Re(\alpha) > 0. Re\,(\beta) \geq 0$. The $(\alpha.\beta)^{th}$ Tempered partial integralof function $f \in L^1[a.b]$, where its defined form is presented as follows:

$$ {}_a^T I_t^{(\alpha.\beta)} f(t) = \left(\Gamma(\alpha)\right)^{-1} \int_a^t (t-u)^{\alpha-1} e^{-\beta(t-u)} f(u)du, \quad t \in [a.b] $$

Definition (2)

We present the formula for defining the reduced partial derivatives of the interval $[a.b]$, and the complex parameters α, $\beta \in IC$, providing that $Re(\alpha). Re(\beta) \geq 0$. The$(\alpha.\beta)^{th}$Tempered partial derivative of function:

$f \in L^1[a.b]$,

$$ {}_a^T D_t^{(\alpha.\beta)} f(t) = \left(\frac{d}{dt} + \beta\right)^n \left({}_a^T I_t^{(n-\alpha.\beta)} f(t)\right). \ t \in [a.b] $$

In 2020, Fernández and Ustaoglu studied the integral and derivative factors of partial calculus and the fractional properties. They examined many analytical properties, discovered many connections and relations used in the classic Riemann-Liouville fractional arithmetic - and then showed how these special relations and the mechanism of fixing parameters with their own values could be used to obtain On special functions, and show the analog of Taylor's theorem for integral inequality to enrich the mathematical Fractional and diffraction and control theory. The fractional differential function equivalent to the Volterra integral equation of the second kin continuous function equation is expressed by the formula:

$$ \begin{cases} D^\alpha \xi(t) = \theta(t, \xi(t)), \ 0 \leq t \leq T, \ \ \xi^{(m)}(0) = \xi_0^{(m)}, \\ m = 0, 1, \ldots, \ s-1, \\ \quad \xi(t) = \sum_{m=0}^{s-1} \frac{t^m}{m!} \xi_0^{(m)} + [\int_0^t \frac{\theta(\zeta, \xi(\zeta))}{(t-\zeta)^{1-\alpha}} d\zeta] / \Gamma(\alpha). \end{cases} $$

This is calculated after fixing the initial values that fit the situation while defining the range of appropriate conditions for the initial arithmetic operations and their properties. This is done through the mathematical formula expressed in the form in the following form:

$$ \xi_h(t_{n+1}) = \sum_{m=0}^{s-1} \frac{t_{n+1}^m}{m!} \xi_0^{(m)} + \theta(t_{n+1}, \xi_h^P(t_{n+1})) h^\alpha / $$

$$ \Gamma(2+\alpha) + \sum_{j=0}^{n} a_{j,n+1} \theta(t_j, \xi_n(t_j)) h^\alpha / \Gamma(2+\alpha), $$

The system of approximate equations of fractal functions obtained using the extended gamma function allows to extend from the study of many phenomena related to stability and partial and complete chaos for many random phenomena that fall within the applications of partial calculus, which many researchers touched upon, note the reference (Matouk *et al.,* 2021), which is the topic The current concern is the concern of many specialists and researchers in the subject, and we leave the room for those wishing to conduct deeper research on this subject so that the values of the regular gamma function are replaced with the values of the extended generalized gamma function.

Generalized Transforms & Some Special Functions

The extended generalized gamma function has been introduced by many researchers since the Kobayashi's generalized gamma function (1991) (Kobayashi, 1991 & Kobayashi, 1991) is a very general function intended to include most known private functions as certain instances, including beta and hypergeometric functions. This was not the only attempt of its kind in the search for a mechanism for using special functions and ways to generalize them, as the researcher proposed 2020 a generalized formula to achieve the same goal in the field of applying random variables in tracking modeling the spread of the Corona 19 pandemic, where the researcher was able through the extended general formula to include some those are special cases, too. The first definition was given by Chaudhry using an approximate series of the generalized gamma function, and at the present time, there are many attempts to generalize the definition and expand the fields of application. The researcher accepted and continues to study the basic special signs and their relationship to known and related transformations with the issue of the extended generalized gamma function. In mathematics, the Mellin transform is integral that can be thought of as a multiple of the two-sided Laplace transform. This transformation was initially named after the Finnish mathematician Hjalmar Milen, who presented it in a paper published in 1897 in Acta Societatis Scientiarum Fennicæ. We mention that this integral transformation is closely related to Dirichlet series theory and is often used in number theory, mathematical statistics, and asymptotic probability expansions theory; In addition to its connection with the Laplace transform, the Fourier transform, the theory of properties of gamma and beta functions and other associated special functions. The two-mile transformation of the function f is expressed by the following mathematical formula (Mellin, 1960),

$$M\{f(\alpha)\} = \varphi(s) = \int_0^{+\infty} x^{\alpha-1} f(x)dx$$

Where $f(x) = (x^k + n)^{-r} e^{-\gamma t}$

The notation used indicates that this is an integral line taken on a vertical line in the complex plane, the real part, as well as the conditions determined by the initial case for calculating the value of the constant C provided that it satisfies certain conditions. The initial conditions under which this inverse transformation exists are established to find the appropriate function formula, which is defined and its validity is known in the inverse Millen transformation theorem and is defined by the following formula:

$$M^{-1}\{\varphi\}(x) = f(x) = \frac{1}{2\pi i}\int_{c-i\infty}^{c+i\infty} x^s \varphi(s)ds$$

Mellin Transform is widely used in software and computer sciences for its use in analyzing complex algorithms, because of its scale stability property, similar to the well-known Fourier transform property. The size of the Fourier transform of the time displacement function and the tracing of the before and dimensional variables is the same as the size of the Fourier transform of the original target function in the trans-formations. This feature is useful for the recognition of images to be studied (Andrews, 1985), as it is easy to scale an object's image easily when the object moves toward or away from the camera smoothly and uncluttered. In the study of quantum mechanics, especially quantum field theory, the Fourier space field is very useful and widely used in tracing states because momentum and position are Fourier transforms for each other, Liam Fitzpatrick and others (Bertrand *et al.,* 1999). Chaudhry was able to study a special case represented by a special transformation of the generalized gamma function proposed by him, where he applied it in some basic fields, which were (thermonuclear astrophysics) as he mentioned in the article, and provided important results that provided essential additions, and here we ask about applications of this transformation of the extended gamma function pro-posed in this article, which is a comprehensive general case for the previous cases, and we leave the space for specialists to provide interest and apply it to related issues and topics with the proposed modifications (Chaudhry, 1998). The Digamma function is defined as the quotient of the derivative of the logarithmic function of the gamma function, and is expressed in the form of the following rational function:

$$\varphi(x) = \frac{d}{dx} ln(\Gamma(x)) = \Gamma'(x)/\Gamma(x)$$

The Digamma function appears in many physical problems, most notably in the comparison and use of inequality-normalizing inequalities, which can be approximated by a general divergent harmonic series. This path was suggested by Abramowitz (Abramowitz and Stegun, 1972), and the Polygamma function is defined by it follows the n+1 derivative differentiation of the gamma function with a simple modification to ex-press the term expressed in the equation below, which is a partial function of the complex numbers ℂ defined as the derivative of the logarithm of the gamma function, and is expressed by the following basic general formula:

$$\varphi^{(n)}(x+1) = \frac{d^{n+1}}{dx^n + 1} ln(\Gamma(x+1)) = \frac{d^{n+1}}{dx^n + 1} ln((x!))$$

Extended Generalized Gamma Function and Some Special General Distributions

The generalized extended gamma distribution is a continuous probability distribution with seven parameters. It is a generalization of the six-parameter gamma distribution. Since many of the distributions commonly used for parametric models in survival analysis (such as generalizations to families of the exponential distribution, Weibull distribution, and extended gamma distribution) are special cases of generalized gamma, they are sometimes used for the parametric case appropriate and representative of a set of mixtures of phenomenological models representing a set Certain field data. This article expresses an opportunity for a new generalization of the generalized gamma distribution called modified extended generalized gamma distribution was introduced to provide greater flexibility in data modeling in practice and fitting data. From the researcher's point of view, the generalization of the proposed new distribution formula includes some of the previous generalizations that were made recently for the general distributions of both gamma and beta. In this way, it presents the totality of the various properties of the proposed distribution in its general form, including the explicit expressions of the moments associated with the parameters of shape, form, and measure, which express the quantities, the starting position of the target states of the data, a basic function for generating instantaneous quasi-random data, the mean deviation, the mean remaining life and the expression of different entropy subjects through the implicit derivative of this distribution. The distribution is able to provide additions, expansions, increases, and decreases in a monotonous manner for different risk ratios in the shape of the bathtub and the shape of the inverted bathtub, and the maximum likelihood estimates for the parameters of this new extended generalization are represented by Mead Mohamed *et al.* (Guseinova and Mamedov, 2007). Several studies have been done on special cases of the generalized gamma distribution (GGD), which includes the four and three-parameters of the general gamma distribution, the Weibull and the two-parameter gamma distributions, and the exponential distribution as special cases by Rama, Shanker; Kamlesh Kumar, Shukla, where during the study the hazard rate function and the random arrangement of the continuous distribution were discussed, and the maximum probabilistic values were discussed to estimate the parameters that express the target condition, and thus the applications of the proposed distribution were discussed with two real-life field data sets and the quality of fit is completely satisfactory and on the generalized distributions of gamma, normal gamma, Whipple and exponential distribution, etc. by Shankar Rama *et al*. (2018). It is noticed from the general formula of the distribution that it is not possible to obtain the unknown parameters in their explicit forms, and that is why they had to be obtained only by solving the nonlinear equations during the process of estimating the parameters. For this, two sets of real data were analyzed to show how the proposed models work and the allowed additions in complex applications, and to show results that allow the new models to be presented in their smooth form. The new extended 8-parameter form of the extended generalized gamma function of a 6-parameter model with the extended generalized gamma distribution model function was introduced by Bachioua, (2009). These proposals show several new distributions as special cases of this proposed distribution, which are given as follows:

$$f_X(\alpha,m,n,p,k,\lambda,\gamma) = \frac{x^{\alpha-1}[x^m+n]^{-r}}{\Lambda_r(\alpha,m,n,p,k,\lambda,\gamma)} exp\{-\lambda x^{p-1} + \gamma x^{1-k}\}$$

The Values of eight parameters of the extended generalized gamma function defined in the domain that makes the integration of the extended gamma function convergent and determinant, which are given as:

$$\Lambda_r(\alpha,m,n,p,k,\lambda,\gamma) = \int_0^{+\infty} x^{\alpha-1}[x^m+n]^{-r}\, exp\{-\lambda x^{p-1} + \gamma x^{1-k}\}dx$$

The new distribution is proposed to cover applications related to the new model function, and it is proposed to extend the study of statistical properties, reliability and severity functions, and estimate the distribution coefficients using different methods, and ensure that this new distribution model is suitable to accommodate diverse applications since it has a variety of shapes, Especially for component lifetime distribution where the presence of offset and intensity parameters is very important where -

$a, b, c \in R^+$.

This proposed model displays a variety of parameters that are characterized by rich dynamics, including sensitivity to chaotic attractants, for example. The general formula for the density distribution has been reformulated in terms of eight basic parameters, in a way that includes and facilitates the process of estimating the parameters, and in order to accommodate their different forms, as some of them were drawn in terms of one parameter with the rest fixed, and many important cases have been shown.

Extended Gamma Distribution Model & Entropy

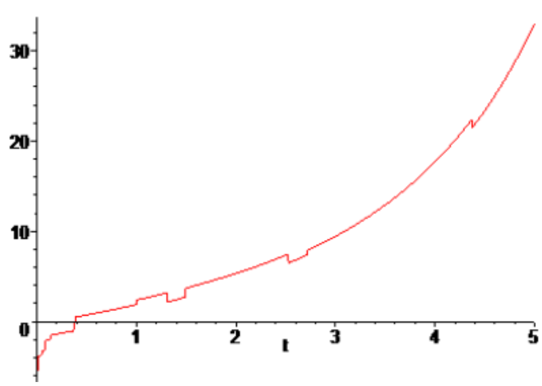
In the last century, German physicist Rudolf Clausius introduced the first concept of entropy in 1850 and was the most important landmark in 19^{th} century physics. The amount of entropy is the work obtained from the desired molecular motion and is a measure of the molecular disorder, or randomness, of the system. The phenomenon of entropy is when a measurement of the thermal energy of a system per unit temperature is not available in order to do useful work for the phenomena to be studied, which is why the idea of entropy provides an intuitive mathematical method for tracking impossible, unstable processes, even though it adheres to the terms of the basic law of conserved energy by Rudolf Clausius, (1865). In 1948, Claude Shannon introduced the concept of information entropy in the field of information theory as the intermediate level inherent in the possible outcomes of a variable by Shannon (Shannon, 1948).

$$\begin{aligned} \mathrm{H}_r(\alpha,m,m,p,k,\lambda,\gamma)(X) &= E[-\ln(f(X)] \\ &= E(\Lambda_r(\alpha,m,m,p,k,\lambda,\gamma)-(\alpha-1)ln(X)- \\ &\quad rln[X^m+n]+\lambda X^{p-1}+\gamma X^{1-k}) \\ &= \Lambda_r(\alpha,m,m,p,k,\lambda,\gamma)-(\alpha-1)E(ln(X)) \\ &\quad -rE(ln[X^m+n]) \\ &\quad +E(ln[\lambda X^{p-1}+\gamma X^{1-k}]) \end{aligned}$$

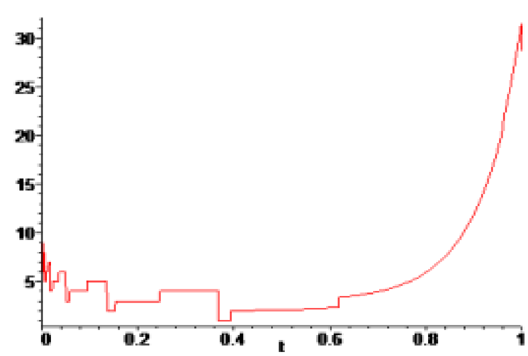
The principle of using entropy theory allows to derive properties for more general distributions in their general form, and by using different constraints and conditions, whereby during fixation of some parameters some aspects are fixed and additional for-mal aspects are traced (Singh, 1998), where the general theoretical idea is summarized by entropy maximization according to the principle of maximum entropy, fixed concerned and unconcerned distribution parameters are defined, and applied to the observed data is a pre-determined set under the constraints and conditions that are within the scope of the underlying data accepted by parameter coverage. This has been applied in phenomena such as flood frequency analysis using observed datasets in order to assess the predictive power of the extended gamma distribution and comparison of flood values by designing different years based on the distributions originating from different families of distributions given by Lu Chen *et al.* (2017).

Tracing the entropy state of random phenomena included in the representative system by extending the generalization of the proposed gamma distribution needs to study the effect of the parameters of the function $\mathrm{H_r}(\alpha, m, m, p, k, \lambda, \gamma)$ by fixing some limits of the values of the eight parameters and studying the variable, which are many thus, the field expands and gives a wide range and margin to represent many of the basic phenomena related to the changes of the studied phenomena targeted by the entropy. **Fig.** (**b:1**), (**b:2**), (**b:3**), (**b:4**), (**b:**5), (**b:6**), (**b:7**), (**b:8**), (**b:9**), (**b:10**) give a clear idea, which are drawn in the plan dimension as follows:

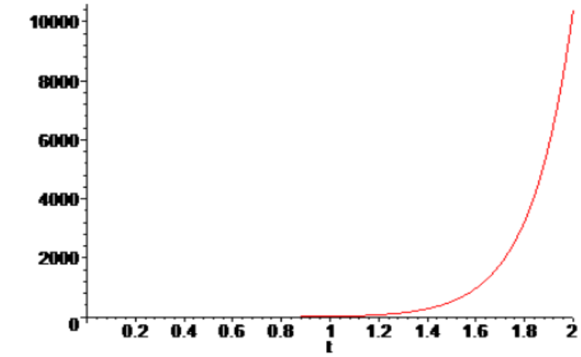
Through the special cases of some different shapes obtained by fixing some parameters of the entropy function defined in terms of the eight parameters, in which the entropy form appears in the form of discontinuous bumps (see graphs (b; 1), (b; 2), (b; 3)), and it expresses a case study that needs to be expanded and to determine its causes & consequences, which are caused by some changes in the values of some parameters and the stabilization of others.



Graph (b;1): Extended Gamma function
$\alpha=1/2; m=2, n=1, p=2, k=1/2, r=1, \lambda=1, \gamma=2$

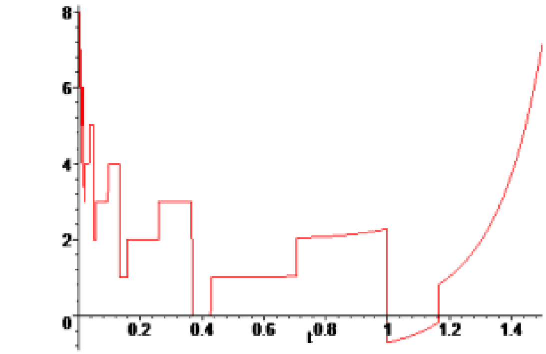


Graph (b;2): Extended Gamma function
$\alpha=4; m=1, n=1, p=4, k=-1, r=-4, \lambda=1, \gamma=2$



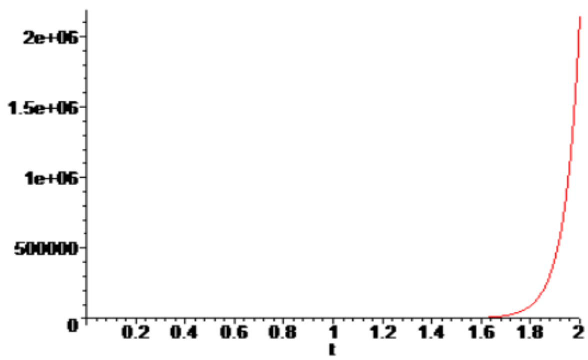
Graph (b;3): Extended Gamma function

$\alpha = 3; m = 2, n = 2, p = 2, k = 1/2, r = 1, \lambda = 5, \gamma = 1$



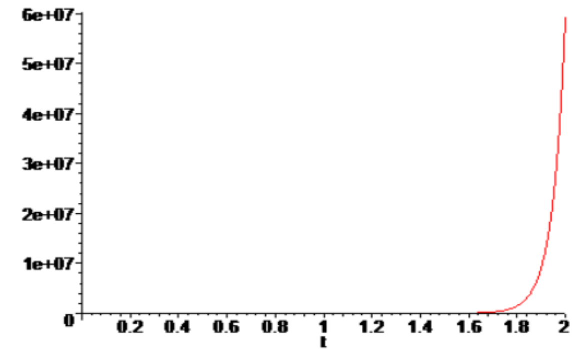
Graph (b;4): Extended Gamma function

$\alpha = 4; m = 3, n = 4, p = 0, k = -1, r = 1, \lambda = 0, \gamma = 2$



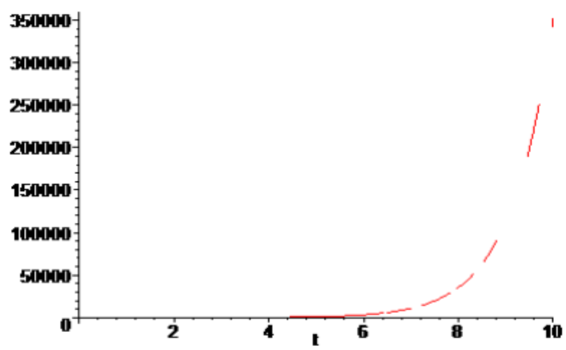
Graph (b;5): Extended Gamma function

$\alpha = 0; m = 2, n = 0, p = 3, k = -1, r = -1, \lambda = 3, \gamma = 1$



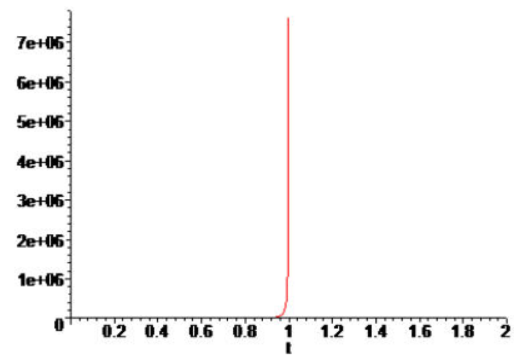
Graph (b;6): Extended Gamma function

$\alpha = 5; m = 1, n = 0, p = 3, k = -1, r = -3, \lambda = 3, \gamma = 1$



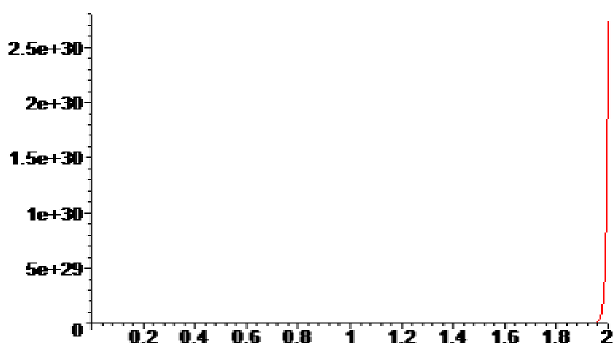
Graph (b;7): Extended Gamma function

$\alpha = 1/4; m = 2, n = -1, p = 2, k = -2, r = -1, \lambda = 1, \gamma = 0$



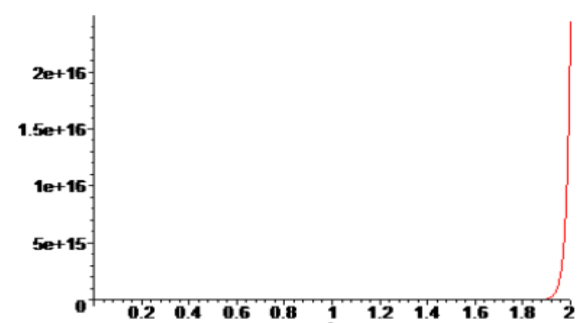
Graph (b;8): Extended Gamma function

$\alpha = 1/4; m = 2, n = -1, p = 2, k = -1, r = -1, \lambda = 2, \gamma = 3$



Graph (b;9): Extended Gamma function

$\alpha = 3; m = 2, n = -2, p = 2, k = -3, r = -2, \lambda = 3, \gamma = 4$



Graph (b;10): Extended Gamma function

$\alpha = 1; m = 1, n = -1, p = 1, k = -2, r = 2, \lambda = 4, \gamma = 5$

Generating Extended Generalized Gamma-Distributed Random Variables

We know the distributive function through the following integrative mathematical formula:

$$F_X(\alpha,m,n,p,k,\lambda,\gamma) = \int_0^x \frac{t^{\alpha-1}[t^m+n]^{-r}}{\Lambda_r(\alpha,m,n,p,k,\lambda,\gamma)} exp\{-\lambda t^{p-1} + \gamma t^{1-k}\}dt$$

This basic function of the function defined by the extended gamma distribution allows us to extend well and continuous and fulfills a one-to-one function, thus accepting the inverse function, allowing us to find the value of the random variable X in terms of the values of the regular random variable in the period [0,1], using the function method Inverse defined by the closed formula, the alignment of the values of the random variable can be completed $U = F_X(\alpha,m,n,p,k,\lambda,\gamma)$, & approximations are made in complex cases Using the closed-form of fitting by Tukey's Lambda distribution models, which is compatible with such problems, especially when using estimation methods that achieve a high degree of accuracy. The process of generating pseudorandom numbers allows us to easily in some special cases, where simpler methods can be used and allow building algorithms compatible with the main goal of using the matching and simulation method for the process of matching and experimenting with non-random numbers and benefiting from the possible results from computer experiments.

DISCUSSION:

The proposed generalization of the extended generalized gamma function and distribution is a commonly used distribution that is appropriate for different age's data, survival data, hydrological data, etc., so it provides more flexible gamma distributions for these different applications. In this family fall kinds of extended generalized gamma function subclasses using different quantitative functions of standardized value distributions, exponential, logistic, logistic, extreme, and various other properties. Each of these subfamilies allows including moments, patterns, entropy, and deviation from the mean and the deviation from the median, and this indicates the new generalized gamma distributions. The extended gamma function is a very flexible function as it relates to real and quasi-real world data, which will encourage these suggestions for in-depth study in the future, especially in the presence of highly effective and efficient computers and software, which allow comparing the results obtained with other different types of distributions used, and frequently in the representation of random phenomena. In this paper, we proposed the extended function and extension by extension, and we defined a new model as an extension of the generalized gamma distribution introduced by Stacy, (1962), and subsequent models and the distributions they include.

CONCLUSION:

The proposed model expands greatly to include several groups of special models, which provide some structural properties of the proposed functions and distributions. We build through the foregoing conclusion on the maximum probability by finding comprehensive families. We also define hybrid models that represent groups of hybrid families used for data analysis. The mixture model represented by this distribution can be considered a model of competitive causes, measurement, & tracking, where the mechanism of activation of causes is controlled by one of the variables that express it and emerge from the studied model through the process of alignment and simulation. We apply the new models to two real data sets to demonstrate their potential and also to perform an effective comparison. It is recommended that simulation and alignment studies be based on the proposed model to verify some characteristics of the finite sample for maximum probability estimates and to study new mechanisms for tracking random phenomena.

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CONFLICTS OF INTEREST:

The authors declare that this article's content has no conflict of interest.

REFERENCES:

1) Abdul Moiz Mohammed and Bachioua Lahcene, (2017). "On Extension of Euler's Beta Function", *Journal of Applied Mathematics & Bioinformatics*, **7**(3), pp 1-11. http://www.scienpress.com/Upload/JAMB/Vol%207_3_1.pdf
2) Abramowitz, M. and Stegun, I. A. (1972). "Laplace Transforms", Ch. 29 in Handbook of Mathematical Functions with Formulas, Graphs, and Mathematical Tables, 9th printing. New York: Dover, pp. 1019-1030.
3) Abramowitz, M.; and Stegun, I. A. (1972). "Digamma function", Handbook of Mathematical Functions with Formulas, Graphs, and Mathematical, section 6.3, Tables (10th ed.) *New York: Dover*, pp. 258–259.
4) Agarwal. S. K. and Kalla. S. K. (1996). "A Generalized Gamma Distribution & its Application in Reliability", *Journal of Commun. Statist, TheoryMeth*., **25**(1), pp.201-210. https://doi.10.1080/03610929608831688
5) Ahmed E., Matouk, and *et al*. (2021). "Complex Dynamics of Some Models of Anti-microbial Resistance on Complex Networks", *Math Meth Appl Sci*., **44**(2), pp.1896-1912. http://dx.doi.org/10.1002/mma.6889
6) Ali, I., S. L. Kalla & H. G. Khajah, (2001). "A Generalized Inverse Gaussian Distribution With τ-confluent Hypergeometric Function", *Int. Transform. Spec. Funct*., **12**(2), pp. 101-114.
7) Al-Musallam, F. and Kalla. S. L. (1997). "Assymptotic Expansions for Generalized Gamma and Incomplete Gamma Functions", *Appl. Anal*., **66**, pp. 173-187.
8) Al-Musallam, F. and S. Kalla, (1998). "Further Results on a Generalized Gamma Function occurring in Diffraction Theory", *Int. Transform. Spec. Funct*., **7**, pp. 175-190.
9) Al-Zamel, A. (2001). "On Generalized Gamma-type Distribution τ-confluent Hypergeometric Function", *Kuwait J. Sci. Engrg*., **28**, pp. 25-36.
10) Andrews, L. C., (1985). "Special Functions for Engineers and Applied Mathematicians", New York: *MacMillan USA*.
11) Andrews. L. C. (1985). "Special Functions of Mathematics for Engineers", 2nd edition, Mc Graw-Hill, Inc., New York, USA. Erdelyi A. (ed). (1953). "Higher Transcendental Functions", Vol. **I**, *McGraw-Hill, New York*, USA.
12) Bachioua Lahcene and Abdulmoiz Mohammed, (2017). "On a Five and Six Parameters Generalization of the Gamma Function", *Theoretical Mathematics & Applications*, **7**(1), pp 1-14.
13) Bachioua, L. (2004). "On Extended and Reliability General Mixture Gamma Distribution Model", A Dissertation Submitted to The College of Science, University of Baghdad in Partial Fulfillment of the Requirements for The Degree of PhD of Science in Mathematics, University of Baghdad, Iraq.
14) Bachioua, L. (2006). "On Generalized Gamma Distribution Function", *First conference in Mathematics*, Dept. of mathematics, college of *Applied science and Mathematics Zarqa Private University Amman*; 18-20 April, Jordan, pp. 406-416.
15) Bachioua, Lahcene, (2009). "On Extended Generalized Gamma Distribution", *International Journal of Applied Mathematics and Statistics*, **150**(D09), pp.98-110. http://www.ceser.in/ceserp/index.php/ijamas/article/view/537
16) Bachioua, Lahcene, (2021). "Some Characterizations of the Extended Beta and Gamma Functions: Properties and Applications", *Int. J. Mat. Math. Sci*., **3**(5), pp. 101-112 https://doi.org/10.34104/ijmms.021.01010112
17) Baleanu. D, and Fernandez. A. (2018). "On Some New Properties of Fractional Derivatives with Mittag-Lefflerkernel", *Communications in Nonlinear Science and Numerical Simulation*, **59**, pp. 444–462.
18) Bertrand J, Bertrand P, and Ovarlez J.-P. (1999). "The Mellin Transform," in the Transforms and Applications Handbook, A. D. Poularikas, Ed., 2nd ed. Boca Raton, *FL: CRC Press*, Ch. 11.
19) Caputo. M., (1967). "Linear Models of Dissipation Whose Q is Almost Frequency Inde-

pendent-II", *Geophys. J. R. Astron. Soc.*, **13**, pp. 529-539.

20) Chaudhry M. Aslam and S. M. Zubair, (2001). "Extended Gamma and Digamma Functions", *Fractional Calculus & Applied Analysis*, **4**(3), pp.303-326.
21) Chaudhry M. Aslam, (1998). "Transformation of the Extended Gamma Function Γ2, 0 0, 2[(b, x)] with Applications to Astrophysical Ther-monuclear Functions", No. (9). https://core.ac.uk/download/pdf/25249162.pdf
22) Caputo. M. (1967). "Linear Models of Dissipation Whose Q is Almost Frequency Independent-II", *Geophys. J. R. Astron. Soc.*, **13**, 529-539.
23) Davis. P. J. (1959). "Leonhard Euler's Integral: a Historical Profile of the Gamma Function", *American. Mathematics*. **66**, pp. 849-869. http://sgpwe.izt.uam.mx/files/users/uami/jdf/proyectos/Euler_integral.pdf
24) Dotsenko. M. (1991). "On Some Applications of Wright's Hypergeometric Function", *Comp. Rend. De I Aead. Bnlgare des sci.*, **44**, pp.13–16.
25) Dugowson S. (1994). "Les Differentielles Metaphysiques: Histoire et Philosophie de la Generalisation de L'ordre de D´erivation", PhD Thesis, Universite Paris Nord, France.
26) Fernandez Arran, and Ustaoglu Ceren, (2020). "On Some Analytic Properties of Tempered Fractional Calculus", *Journal of Computational and Applied Mathematics*, **366**(112400), pp.1-16. https://doi.org/10.1016/j.cam.2019.112400
27) Galue, L., S. Alobaidi and S. L. Kalla, (2001). "Further Generalizations of Kobashi's Gamma Function", *Serdica Math. J.*, **27**, pp. 303-316.
28) George B. Arfken, and Hans J. Weber, "Mathematical Methods for Physicists", 16th Ed., Sabre Foundation, *Elsevier*, pp. 511, USA.
29) Guseinova. I. I., and Mamedov B. A. (2007). "Unified Treatment for the Evaluation of Generalized Complete and Incomplete Gamma Functions", *Journal of Computational and Applied Mathematics*, **202**, pp. 435 – 439.
30) Jade, I. G. (1953). "The Population Frequencies of Species and the Estimation of Population Parameters", *Biometrika*, **40**, pp. 237-260.
31) Jarad F., Abdeljawad T., & Alzabut J. (2018). "Generalized Fractional Derivatives Generated by a Class of Local Proportional Derivatives", *European Physical Journal Special Topics*, **22**, pp. 3457–3471.
32) Kobayashi, K. (1991). "On a Generalized Gamma Function Occurring in Diffraction Theory", *Journal of Physical Society of Japan*, **60**, pp. 1501-1512.
33) Kobayashi, K. (1991). "Plane Way Diffraction by a Strip", Exact and asymptotic solutions. *J. of Physical Society of Japan*, **60**, pp. 1891-1905. https://doi.org/10.1143/JPSJ.60.1891
34) Larry, C. A. (1992). "Special Functions of Mathematics for Engineers", *McGraw-Hill, New York, USA*.
35) Lebedev. N. (1972). "Special Functions and Their Applications", *Dover Publications Inc.*, New York, USA.
36) Liam Fitzpatrick A, Jared Kaplan, and Balt C. van Rees, "A Natural Language for AdS/CFT Correlators". https://en.wikipedia.org/wiki/Mellin_transform#cite_note-9
37) Lu Chen, Vijay P. Singh, and Feng Xiong, (2017). "An Entropy-Based Generalized Gamma Distribution for Flood Frequency Analysis", *Entropy*, **19**(6), pp. 239 -248. https://doi.org/10.3390/e19060239
38) Malovichko. V. (1976). "On a Generalized Hypergeometric Function and Some Integral Operators", *Math. Phy*, **19**, pp. 99–103.
39) Mead Mohamed, Mazen Nassar, and Dey Sanku, (2018). "A Generalization of Generalized Gamma Distributions", *Pak. j. stat. oper. res*. **14**(1), pp. 121-138. https://doi.org/10.18187/pjsor.v14i1.1692
40) Mellin, H J. (1960). "Zur Theorie Zweier Allgemeinen Klassen Bestimmter Integrale", *Acta Societatis Scientiarum Fennicæ*, **2**, pp. 1–75.
41) Niazai S, Rahimzai AA, Danesh M, and Safi B. (2022). Numerical solution of diffusion equation with caputo time fractional derivatives

using finite-difference method with Neumann and Robin boundary conditions, *Int. J. Mat. Math. Sci*., **4**(1), 01-14. https://doi.org/10.34104/ijmms.022.010014

42) Noble B. (1988). "Methods Based on the Wiener-Hope Technique for Solution of Partial Differential Equations", *Chelsea, New York, USA*.
43) Rainville, E. D. (1960)."Special Function", the Macmillan Company, New York.
44) Rudolf Clausius, (1865). "The Mechanical Theory of Heat with its Applications to the Steam Engine and to Physical Properties of Bodies", London, UK.
45) Saboor, A. and M. Ahmad, (2012)."Bivariate Generalized Gamma-type Probability Function Using a Hypergeometric Function of two Variables", *Pak. J. Statist*., **28**, pp. 81-91.
46) Saboor, A., Provost S. B. and Ahmad. M. (2012). "The Moment Generating Function of a Bivariate Gamma-type Distribution", *Appl. Math. Comput*., **218**(24), pp. 11911-1192.
47) Dugowson S. (1994). "Fractional Integrals and Derivatives: Theory and Applications", *Taylor & Francis*, London, UK.
48) Sandifer, C. Edward, (2007). "The Early Mathematics of Leonhard Euler", Mathematical Association of America, Washington, DC, 91, USA.
49) Saxena, R. K. and S. L. Kalla, (2001). "On a Generalized Gamma Function Occurring in Diffraction Theory", *Int. J. Appl. Math*., **5**, pp. 189-202.
50) Shanker Rama, and Shukla Kamlesh Kumar, (2018). "A Generalization of Generalized Gamma Distribution", *International J. of Computational and Theoretical Statistics*, **1**(6), pp.1-12. http://dx.doi.org/10.12785/ijcts/060105
51) Shannon, Claude E. (1948). "A Mathematical Theory of Communication", *Bell System Technical Journal*, **27**(3), pp. 379–423. http://dx.doi.org/10.1002/j.1538-7305.1948.tb01338.x
52) Singh, V. P. (1998). "Entropy- Based Parameter Estimation in Hydrology", *Springer*, Dordrecht, The Netherlands.
53) Stacy, EW. (1962). "A Generalization of the Gamma Distribution", *The Annals of Mathematical Statistics*, **33**, pp.1187-1192. http://dx.doi.org/10.1214/aoms/1177704481
54) Wikipedia. (2015)."Adrien-Marie Legendre". https://en.wikipedia.org/wiki/Adrien-Marie_Legendre
55) Yousif AAA., and Arbab SY. (2022). A numerical investigation of the structural performance of double-layered grid domes using software packages, *Int. J. Mat. Math. Sci*., **4**(2), 35-44. https://doi.org/10.34104/ijmms.022.035044

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