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Application of Perturbation Method to Approximate the Solutions of Differential Equation

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ABSTRACT

We investigate the core features of the perturbation method with the help of some simple but sophisticated problems and demonstrate how much accurately it predicts the solutions of the problems. To fulfill the target, we use the method for getting the solution of differential equations with initial and boundary conditions. Then the results obtained are compared with the series solution and the exact/numerical solution by using Mathematica and Fortran Programming. The comparisons are shown graphically. Also, the perturbation series approximation and the exact or numerical solution are in good agreement. Our investigation shows that a certain number of terms of the perturbation series give an excellent approximation than the same number of terms of the numerical solution.

Keywords: Perturbation method, Approximation, Houndary conditions, and Neumann boundary condition.

INTRODUCTION:

Perturbation theory developed by Taiwo and Osilagum, (2011) is widely used for the simplification of complex mathematical problems. The use of perturbation theory will allow approximate solutions to determine the problems that cannot be solved by traditional analytical methods. However, in many cases, real-life situations can require much more difficult mathematical models, such as non-linear differential equations.

Perturbation theory is a powerful mathematical technique used to approximate solutions to problems that are difficult to solve exactly. It is particularly useful in physics, chemistry, engineering, and applied mathematics, where many complex systems can be represented by differential equations that are challenging to solve analytically. Perturbation theory provides a systematic way to obtain approximate solutions by breaking down the problem into simpler, more solvable parts which was investigated by Debwan and Hasan, (2020). The basic idea behind perturbation theory is to start with a known, easily solvable problem (the unperturbed problem) and then introduce small corrections or perturbations to it. These perturbbations are typically represented by small parameters that quantify the deviation from the idealized, unperturbed system. By treating these perturbations as small deviations from the known solution, one can develop a series expansion in terms of these parameters, which can then be used to derive increasingly accurate approximations to the true solution. There are two main types of perturbation theory: regular perturbation theory and singular perturbation theory. Regular perturbation theory developed by Dehghan and Shakeri, (2008) in which the perturbations are assumed to be small across the entire domain of interest. This allows for the development of a systematic series expansion, such as a power series, in terms of the small parameter. Each term in the series represents a higher-order correction to the solution, and the series can often be truncated at a certain order to obtain a sufficiently accurate approximation. The other is singular perturbation theory; singular perturbation theory is used when the perturbations are not small throughout the entire domain, but rather become significant in certain regions or under certain conditions. In such cases, a straightforward series expansion may not converge, and alternative techniques, such as matched asymptotic expansions or boundary layer analysis, are employed to obtain accurate solutions near the points of interest. Once the equation is non-dimensionalized, perturbation theory requires taking advantage of a "small" parameter that appears in an equation was explained by (Shakeri and Dehghon 2008; Elrazeg *et al.,* 2022). This parameter, usually denoted "ε" is on the order of $0 < \varepsilon << 1$.

Perturbation theory has its roots in early celestial mechanics, where the theory of epicycles was used to make small corrections to the predicted paths of planets. Perturbation methods are powerful techniques used in mathematics and physics to approximate the results to differential equations, especially when exact solutions are difficult or impossible to find. These methods are particularly useful for problems where a small parameter exists, allowing for an expansion around a known solution. Perturbation theory relies on the existence of a small parameter, often denoted as ϵ, which quantifies the deviation from a simpler problem. This parameter could represent a physical quantity like mass ratio, coupling constant or geometric scale. The central idea of perturbation theory is to approximate the solution to a complex problem by iteratively correcting a known solution to a simpler problem. This known solution is usually obtained by setting the small parameter to zero. Perturbation theory often involves analyzing the behavior of the solution as the small parameter approaches zero. Asymptotic behavior provides insights into the dominant terms contributing to the solution. The accuracy and validity of the approximate solution are validated by comparing it with exact solutions (if available), numerical simulations, or experimental data. Perturbation theory finds applications in diverse fields such as celestial mechanics, quantum mechanics, fluid dynamics, population dynamics, and many more. Advanced techniques such as multiple scales method, matched asymptotic expansions, and resummation methods are employed to improve the approximation problems (Nino *et al*., 2013; Gervais *et al*., 1975). Perturbation theory provides a systematic framework for tackling complex problems and gaining insights into the behavior of physical systems. Its versatility and applicability make it an indispensable tool in scientific and engineering research. The beginnings of perturbation theory can be traced to Isaac Newton's work on the gravitational interactions between celestial bodies in the 17th century. While Newton provided exact solutions for two-body problems, the interactions of more than two bodies posed significant challenges. In the late 18th and early 19th centuries, mathematicians such as Joseph-Louis Lagrange and Pierre-Simon Laplace made significant contributions to celestial mechanics. They developed perturbation methods by He, (2003) to study the effects of gravitational interactions among multiple celestial bodies. Aghakhani, (2015) introduced the concept of secular perturbations, which long-term effects are arising from gravitational interactions that cause slow changes in the orbits of celestial bodies over time. The development of series solutions for differential equations by mathematicians like Leonhard Euler and Joseph Fourier provided a mathematical foundation for perturbation methods. Laplace and Macgillivry, (2008) developed the methods for finding asymptotic expansions of integrals, which laid the groundwork for the asymptotic analysis used in perturbation theory. William Rowan Hamilton's reformulation of classical mechanics in terms of Hamiltonian mechanics provided a new framework for perturbation theory. Wilsen and Rallison, (2007) made significant contributions to perturbation theory in celestial mechanics, introducing the concept of canonical transformations and developing perturbation methods to study the stability of the solar system. Quantum Mechanics: Perturbation theory found widespread application in quantum mechanics, particularly in the development of quantum electrodynamics (QED) by physicists such as Paul Dirac, Richard Feynman, Julian Schwinger, and Sin-Itiro Tomonaga. Feynman diagrams revolutionized perturbation theory in quantum field theory, providing a graphical representation of particle inter-actions and facilitating calculations of scattering amplitudes.

Perturbation methods have been extended by Xia and Zhang, (2024) to study nonlinear dynamical systems, chaos theory, and bifurcation theory. Perturbation techniques are widely used in engineering disciplines such as fluid dynamics, structural mechanics, and control theory to analyze the effects of small disturbances on system behavior. Throughout its history, perturbation theory has evolved from its origins in celestial mechanics to become a fundamental tool in various branches of science and engineering. Its development has been driven by the need to understand and quantify the effects of small perturbations on complex systems, leading to advancements in both theory and application. Overall, the development of basic perturbation theory by Clenshaw and Norton, (1963) involves a systematic process of approximating solutions to differential equations by expanding them in powers of a small parameter and iteratively refining the solution to higher orders of accuracy. These methods provide valuable insights into complex systems and phenomena that cannot be fully understood using exact analytical techniques.

METHODOLGY:

The methodology of perturbation methods involves a systematic approach to approximate results to differential equations by treating small deviations or perturbations from a known, easily solvable problem. Here's a general outline of the methodology. There are three steps and eight procedures of perturbation analysis. The three steps are given below;

1) To transform the main problem into a perturbation problem by taking a small parameter δ.
2) To consider an expression for the solution in the form of a perturbation series and determine the coefficient of that series.
3) To regain the solution to the main problem by adding the perturbation series for the appropriate value of δ.

Step (1): There is sometimes ambiguity because there are many ways to introduce an δ. However, it is preferable to introduce δ in such a way that the zeroth-order solution i.e. the leading term in the perturbation series is obtainable as a closed-form analytic expression.

Step (2): By setting δ =0 in the perturbation problem, a first-order solution consists of finding the first two terms in the perturbation series, and so on.

Step (3): Begin by identifying the differential equation that describes the problem of interest. This equation typically represents the behavior of a system under consideration, such as a physical system governed by the laws of physics or an engineering system described by mathematical models.

Now the eight procedures are as follows;

Introduce Perturbation Parameters

Identify small parameters that quantify the deviations or perturbations from the idealized, unperturbed system. These parameters may represent small variations in system parameters, initial conditions, boundary conditions, or external influences.

Decompose the Solution

Decompose the solution into two parts: the outcome of the unperturbed problem and the perturbation correction terms. The unperturbed solution represents the solution to a simplified version of the problem that is easily solvable, while the perturbation correction terms account for the effects of small deviations from this idealized solution.

Expand the Solution in a Series

Express the solution as a series expansion in terms of the small perturbation parameters. This series expansion typically takes the form of a power series, where each term represents a higher-order correction to the solution. The coefficients of the series are functions of the perturbation parameters and are determined iteratively.

Derive Equations for the Coefficients

Substitute the series expansion into the original differential equation and equate coefficients of like powers of the perturbation parameters on both sides of the equation. This yields a set of equations for the coefficients of the series expansion, which can be solved order by order to obtain increasingly accurate approximations to the solution.

Iterative Solution Procedure

Solve the equations for the coefficients iteratively, starting from the zeroth-order term corresponding to the unperturbed solution and proceeding to higher-order terms. Each successive iteration yields a more accurate approximation to the true solution by incorporating additional corrections from higher-order perturbation terms.

Truncate the Series Expansion

Truncate the series expansion at a certain order based on the desired level of accuracy or practical considerations. In many cases, only a few terms of the series are needed to obtain sufficiently accurate approximations, especially if the perturbation parameters are small.

Analyze Convergence and Validity

Analyze the convergence properties of the series expansion and assess the validity of the perturbation approach. Ensure that the perturbation parameters are indeed small and that the series converges to the true solution within the desired range of validity.

Interpretation and Application

Interpret the results into the domain and apply the approximate solution to analyze the behavior of the system, make predictions, or design engineering solutions. Understand the obstacles of the perturbation approach and validate the results through comparisons with numerical simulations or experimental data when possible. By following these steps and procedures, perturbation methods provide a systematic and powerful framework for approximating solutions to differential equations in a wide range of scientific and engineering applications, allowing researchers and engineers to tackle complex problems that are otherwise difficult to solve analytically.

Non-singular Perturbation Theory with First-order

First-order non-singular perturbation theory introduced by Nayfeh, (1981) is a technique used to approximate solutions to differential equations where perturbations become significant at certain points or under specific conditions, but are not small throughout the entire domain. This method was employed by Kumar and Parul, (2011) for the straightforward application of regular perturbation theory, which assumes small perturbations across the entire domain, is inadequate. Now, we have to solve the following differential equation,

$$Dg(x) = \lambda g(x)\,, \tag{1}$$

Where, D implies differential operator, and λ is an eigenvalue. Now it can be written as,

$$D = D^{(0)} + \varepsilon D^{(1)}\,, \tag{2}$$

Where, ε is very small, and operator $D^{(0)}$ are known. That is, one has a set of solutions $f_n^{(0)}(x)$, labeled by index n, such that

$$D^{(0)} f_n{}^{(0)}(x) = \lambda_n{}^{(0)} f_n{}^{(0)}(x)\,. \tag{3}$$

Furthermore, one assumes an orthonormal set,

$$\int f_m{}^{(0)}(x) f_n{}^{(0)}(x)dx = \delta_{mn}\,, \tag{4}$$

Where, δ_{mn} implies the Kronecker delta. Now, for the unperturbed solutions $f_n^{(0)}(x)$. That is,

$$g(x) = f_n^{(0)}(x) + \mathbb{O}(\varepsilon). \tag{5}$$

And $\lambda = \lambda_n^{(0)} + +\mathbb{O}(\varepsilon)$, (6)

Where, $\mathbb{O}$ denotes by big-O pattern, of the perturbation. We consider the linear combination $f_n^{(0)}(x)$:

$$g(x) = \sum_m c_m f_m{}^{(0)}(x)\,, \tag{7}$$

for $c_m = \mathbb{O}(\varepsilon)$except for n, where $c_n = \mathbb{O}(1)$ by orthogonality condition,

$$c_n \lambda_n^{(0)} + \varepsilon \sum_m c_m \int f_n^{(0)}(x) D^{(1)} f_m^{(0)}(x) dx = \lambda c_n \ , \tag{8}$$

Where

$$\sum_m A_{nm} c_m = \lambda c_n \ , \tag{9}$$

where the matrix elements A_{nm} are given by

$$A_{nm} = \delta_{nm} \lambda_n^{(0)} + \in \int f_n^{(0)}(x) D^{(1)} f_m^{(0)}(x) dx \ . \tag{10}$$

The trivial solution,

$$\lambda = \lambda_n^{(0)} + \varepsilon \int f_n^{(0)}(x) D^{(1)} f_n^{(0)}(x) dx \ . \tag{11}$$

By order $\mathbb{O}(\varepsilon^2)$, we get

$$g(x) = f_n^{(0)}(x) + \varepsilon f_n^{(1)}(x) \ . \tag{12}$$

So that

$$(D^{(0)} + \varepsilon D^{(1)})(f_n^{(0)}(x) + \varepsilon f_n^{(1)}(x)) = (\lambda_n^{(0)} + \varepsilon \lambda_n^{(1)})(f_n^{(0)}(x) + \varepsilon f_n^{(1)}(x)) \ , \tag{13}$$

gives an equation for $f_n^{(1)}(x)$,

$$\delta(x-y) = \sum_n f_n^{(0)}(x) f_n^{(0)}(y) \ . \tag{14}$$

To give $f_n^{(1)}(x) = \sum_{m(\neq n)} \frac{f_m^{(0)}(x)}{\lambda_n^{(0)} - \lambda_m^{(0)}} \int f_m^{(0)}(y) D^{(1)} f_n^{(0)}(y) dy \ .$ (15)

(Salem and Thanoon, 2021; Liu and Chang, 2022) interpreted these results in the domain which can be applied the approximate solution to analyze the behavior of the system, make predictions, or design engineering solutions. Understand the limitations of the first-order non-singular perturbation approach and validate the results through comparisons with numerical simulations or experimental data when possible. By following these steps, first-order non-singular perturbation theory provides a systematic approach to approximate solutions to differential equations in situations where perturbations become significant in specific regions or under certain conditions. This method allows researchers and engineers to tackle complex problems that cannot be explained using regular perturbation theory alone.

Approximate Solution of an Initial-value Problem

The initial value problem was chosen by Fowkes, (1968) and the boundary value problem was chosen by Baltaveva and Agarwal, (2018), which gives the following,

$$y''=f(x)y, \quad y(0) = 1, \quad y'(0) = 1, \tag{16}$$

Where, $f(x)$ is continuous, this problem has no closed-form solution except for very special choices for $f(x)$.

First, we introduced an ε as,

$$y''= \varepsilon f(x)y, \quad y(0) = 1, \quad y'(0) = 1. \tag{17}$$

Secondly, we take $y(x)$ as,

$$y(x) = \sum_{n=0}^{\infty} \varepsilon^n y_n(x). \tag{18}$$

Where, $y_o(0) = y_o^{'}(0) = 1$ and $y_n(0) = y_n^{'}(0) = 0 \quad (n \geq 1)$

The zeroth-order problem $y''=0$ is obtained by setting ε=0. The nth-order problem (n ≥1) is obtained by Eq. (18) into Eq. (17) and setting the coefficient of ε^n (n ≥1) equal to zero. The result is

$$y''_n = f(x)y_{n-1},\ y_n(0)=0,\ y_n'(0)=0. \tag{19}$$

The solution to (19) is

$$y_n = \int_0^x dt \int_0^t f(s)y_{n-1}(s)ds,\ \ n \geq 1. \tag{20}$$

Now, the successive terms in the parturition series (3):

$$y(x) = 1 + x + \varepsilon\int_0^x dt\int_0^t (x+s)f(s)ds + \varepsilon^2\int_0^x dt\int_0^t f(s)ds\ \int_0^s dt\int_0^v (1+u)f(v)dv + \cdots \tag{21}$$

The nth term in this series is by $\varepsilon^N x^{2N} K^N (1+|x|)/(2N)$ with upper bound K for |f(t)| in the internal $0 \leq |t| \leq x|$.

Neumann Boundary Conditions

Neumann boundary conditions developed by Kadum and Abdul-Hassan, (2023) are commonly encountered in partial differential equations (PDEs). They specify the function at the boundary of a domain rather than specifying the function itself. Neumann boundary condition concept arise in various physical and mathematical contexts, particularly in problems involving diffusive processes, heat transfer, fluid dynamics, and electromagnetism was given by Saltzman, (1962).

Consider the differential equation,

$$y'' + y = 0. \tag{22}$$

On the interval $[a, b]$ take the form:

$$y'(a) = \alpha \text{ and } y'(b) = \beta. \tag{23}$$

Where, α and β are given numbers. Now, we have

$$\nabla^2 y + y = 0. \tag{24}$$

Where, ∇^2 denotes the given conditions on a domain $\Omega \subset \mathbb{R}^n$ take the form:

$$\frac{\partial y}{\partial n}(x) = f(x) \quad \forall x \in \partial\Omega. \tag{25}$$

The normal derivative which shows up on the left-hand side is defined as:

$$\frac{\partial y}{\partial n}(x) = \nabla y(x).n(x). \tag{26}$$

Where, ∇ is the gradient (vector) and the dot is the inner product.

The Perturbation Process employ to the outcome of an Algebraic Equation

The Perturbation Method Expanded with proposed equation

Consider the following equation. First, the equation is introduced as seen below.

$x^2 + \varepsilon x - 1 = 0,\ 0 < \varepsilon\varepsilon << 1.$

The Perturbation Technique for Solving Algebraic Equation

Leading Order Solutions

The approximation process to $x^2 + \varepsilon x - 1 = 0$ is to set $\varepsilon = 0$.

This reduces to:

$$x^2 = 1. \tag{27}$$

Or,

$$x = \pm 1. \tag{28}$$

First-order solution

The approximation with second order,

$$x = \pm 1 + \delta(x), \tag{29}$$

Where, $\delta(x)$ is some connection factor.

Inserting $x=\pm 1$ into $x^2+\varepsilon\, x\text{-}1=0$ yields;

$$(1+\delta(x))(1+\delta(x))+\varepsilon(1+\delta(x))-1=0. \quad (30)$$

Expanding (30) $\Rightarrow$

$$\delta(x)^2+2\delta(x)+1+\varepsilon+\varepsilon\delta(x)-1=0. \quad (31)$$

Solving the remainder for (31) yields,

$$\delta(x)=-\frac{\varepsilon}{2}. \quad (32)$$

Substitution of $\delta(x)$ back into Eq. (29) for the positive roots yields:

$$x=1+\left(-\frac{\varepsilon}{2}\right). \quad (33)$$

Second-order solution

The approximation with third result is,

$$x=1+\left(-\frac{\varepsilon}{2}\right)+\beta(x) \quad (34)$$

Putting (34) into $x^2+\varepsilon x-1=0$ yields:

$$\left(1-\frac{\varepsilon}{2}+\beta(x)\right)\left(1-\frac{\varepsilon}{2}+\beta(x)\right)+\varepsilon\left(1-\frac{\varepsilon}{2}+\beta(x)\right)-1=0\cdot \quad (35)$$

Expanding (35) $\Rightarrow$

$$1-\frac{\varepsilon}{2}+\beta(x)-\frac{\varepsilon}{2}+\frac{\varepsilon^2}{4}-\frac{\varepsilon}{2}\beta(x)+\beta(x)-\frac{\varepsilon}{2}\beta(x)+$$

$$\beta(x)^2+\varepsilon-\frac{\varepsilon}{2}+\varepsilon\beta(x)-1=0. \quad (36)$$

The equation (36) is employ for $\beta(x)$,

$$\beta(x)=\frac{\varepsilon^2}{8}. \quad (37)$$

Substitution of $\beta(x)$ back into (34) for yields the third positive root approximation:

$$x=1+\left(-\frac{\varepsilon}{2}\right)+\frac{\varepsilon^2}{8}. \quad (38)$$

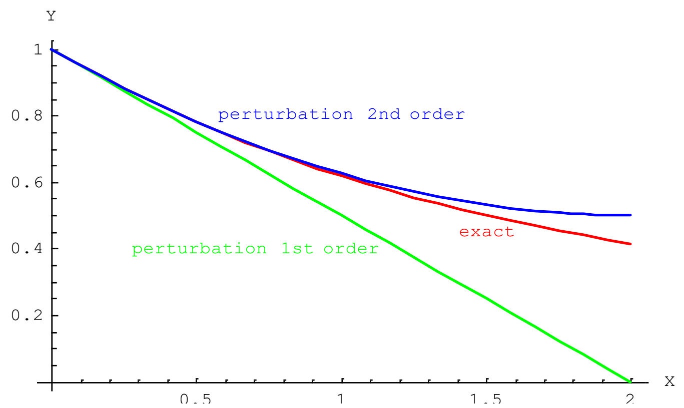

Fig. 1: Graphical comparison between exact and perturbation solution.

Analytic Result

Here, gusting the solution by Shampine, (1968),

$$y=e^{Rx} \quad (39)$$

and Substituting equation $y=e^{Rx}$ into the following equation

$$\varepsilon\frac{d^2y}{dx^2}+2\frac{dy}{dx}+2y=0\cdot$$

Now,

$$\varepsilon R^2 + 2R + 2 = 0. \tag{40}$$

For roots R_1 and R_2,

$$R_1 = \frac{-2+\sqrt{4-8\varepsilon}}{2\varepsilon}. \tag{41}$$

and $$R_2 = \frac{-2-\sqrt{4-8\varepsilon}}{2\varepsilon}. \tag{42}$$

Now,

$$y = C_1 e^{R_1 x} + C_2 e^{R_2 x}. \tag{43}$$

Substitution of equations (41) and (42) into equation (43) yields the following:

$$y = C_1 e^{\frac{-2+\sqrt{4-8\varepsilon}}{2\varepsilon}x} + C_2 e^{\frac{-2-\sqrt{4-8\varepsilon}}{2\varepsilon}x}. \tag{44}$$

Enforcement of the initial condition seen in equation $y(0) = 0$ to Eq. (44) yields:

$$C_1 = C2. \tag{45}$$

Substituting Eq. (45) into (44) Eq. (44) yields,

$$-C_2 e^{\frac{-2+\sqrt{4-8\varepsilon}}{2\varepsilon}} + C_2 e^{\frac{-2-\sqrt{4-8\varepsilon}}{2\varepsilon}} = 1 \tag{46}$$

Solving Eq. (46) for C_2 and then using Eq. (45) to solve for C_1 yields:

$$C_2 = \frac{1}{-e^{\frac{-2+\sqrt{4-8\varepsilon}}{2\varepsilon}} + e^{\frac{-2-\sqrt{4-8\varepsilon}}{2\varepsilon}}} \tag{47}$$

and $$C_1 = \frac{1}{-e^{\frac{-2+\sqrt{4-8\varepsilon}}{2\varepsilon}} + e^{\frac{-2-\sqrt{4-8\varepsilon}}{2\varepsilon}}} \tag{48}$$

Substitution of equations (47) and (48) into equation (44) which gives,

$$y_{analytical} = -\left[\frac{1}{-e^{\frac{-2+\sqrt{4-8\varepsilon}}{2\varepsilon}} + e^{\frac{-2-\sqrt{4-8\varepsilon}}{2\varepsilon}}}\right] e^{\frac{-2+\sqrt{4-8\varepsilon}}{2\varepsilon}x} + \frac{1}{-e^{\frac{-2+\sqrt{4-8\varepsilon}}{2\varepsilon}} + e^{\frac{-2-\sqrt{4-8\varepsilon}}{2\varepsilon}}} e^{\frac{-2+\sqrt{4-8\varepsilon}}{2\varepsilon}x} \tag{49}$$

Comparison of Perturbation Approximation to the Analytical Solution

Let, $\varepsilon = 0.01$, valves determined from equations (41), (42), (47) and (48) the given table,

Table 1: Analytical results obtained for the ordinary differential equation.

Small parameter	Analytical Roots		Analytical Constants	
ε	Root- 1	Root-2	C_1	C_2
0.01	-1.00505	-198.99494	2.73204	-2.733205

Data from the **Table 1**,

$$\% \text{ Error} = \left[\frac{(Perturbation\ valve - Actual\ valve)}{Actual\ value}\right] * 100$$

$$y_{composite} = -ee^{\frac{-2x}{\varepsilon 1}} + ee^{-x} \text{ and } y_{composite} = -\left[\frac{1}{-e^{\frac{-2+\sqrt{4-8\varepsilon}}{2\varepsilon}} + e^{\frac{-2-\sqrt{4-8\varepsilon}}{2\varepsilon}}}\right] e^{\frac{-2+\sqrt{4-8\varepsilon}}{2\varepsilon}} + \frac{1}{-e^{\frac{-2+\sqrt{4-8\varepsilon}}{2\varepsilon}} + e^{\frac{-2-\sqrt{4-8\varepsilon}}{2\varepsilon}}} e^{\frac{-2-\sqrt{4-8\varepsilon}}{2\varepsilon}x}.$$

Table 2: Exact results and Perturbation to the ordinary differential equation.

X	Y Analytical	Y Composite	% Error
0	0	0	-

0.011519	2.424557348	2.415660385	-0.36695
0.023038	641622008	2.629258016	-0.46805
0.034907	2.635230466	2.622507364	-0.48281
0.069813	2.546917667	2.534980398	-0.46869
0.10472	2.4591161	2.448021737	0.45115
0.139626	2.374339022	2.364043877	-0.4336
0.174533	2.292484598	2.282946823	-0.41605
0.20944	2.213452074	2.204631754	-0.39849
0.244346	2.137144166	2.129003234	-0.38093
0.279253	2.063466944	2.055969103	-0.36336
0.314159	1.992329715	1.985440363	-0.34579
0.349066	1.923644915	1.917331067	-0.32822
0.383972	1.857327997	1.851558218	-0.31065
0.418879	1.79329733	1.788041666	-0.29307
0.453786	1.731474094	1.726704009	-0.27549
0.488692	1.671782192	1.667470503	-0.25791
0.523599	1.614148144	1.610268966	-0.24032
0.558505	1.558501008	1.555029691	-0.22273
0.593412	1.504772286	1.501685365	-0.20514
0.628319	1.452895841	1.450170984	-0.18755
0.663225	1.402807816	1.400423771	-0.16995
0.698132	1.354446556	1.352383105	-0.15235
0.733038	1.307752533	1.305990444	-0.13474
0.767945	1.262668268	1.261189255	-0.11713
0.802851	1.219138265	1.217924943	-0.09952
0.837758	1.177108943	1.176144786	-0.08191
0.872665	1.136528565	1.135797871	-0.06429
0.907571	1.09734718	1.096835032	-0.04667
0.942478	1.059516559	1.059208789	-0.02905
0.977384	1.022990133	1.022873291	-0.01142
1.012291	0.987722942	0.987784259	0.006208
1.047198	0.953671574	0.953898935	0.023841
1.082104	0.920794114	0.921176026	0.041476
1.117011	0.889050091	0.889575656	0.059115
1.151917	0.858400431	0.859059317	0.076757
1.186824	0.828807407	0.829589821	0.094402

We notice that the total outcome is rapidly reduced as x increases.

For the linear ordinary differential equation can be noticed in **Fig. 2**.

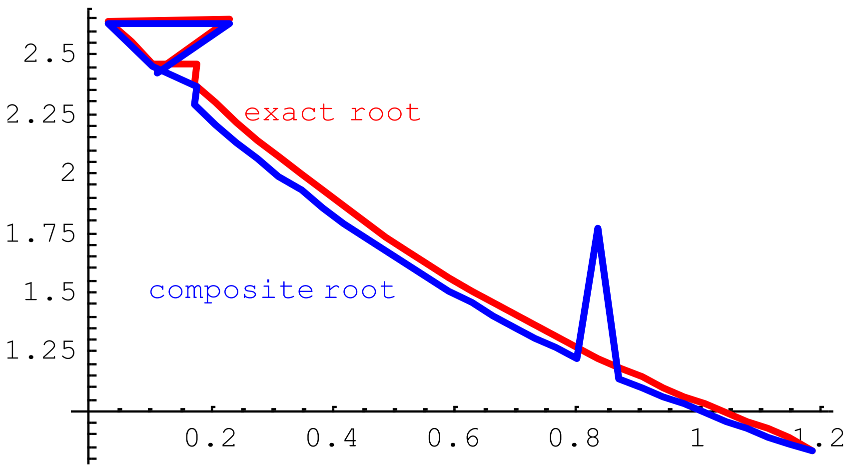

Fig. 2: Comparative solutions draw for the proposed equation.

For the linear ordinary differential equation can be noticed with approximations percent error in **Fig. 3**.

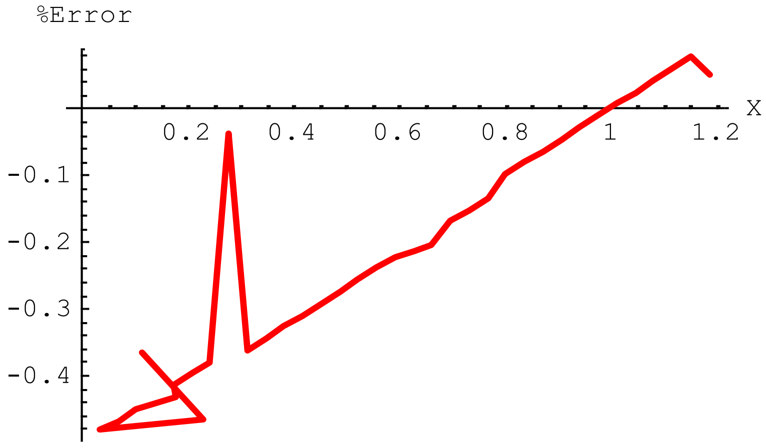

Fig. 3: The percent error of regular perturbation graph for the proposed equation.

Approximate Solutions of an IVP using the Perturbation Method

The most important and excellent example of perturbation in this paper is example 1. It can be taken as a notable example of the perturbation solution of given equation. It shows how much good the perturbation solution is!

Example 1

If $\varepsilon \ll 1$, obtain the perturbed equation from

$y'' - \varepsilon xy = 0, y(0) = 1, y'(0) = 1$

Solution: Given,

$y'' - \varepsilon xy = 0, y(0) = 1, y'(0) = 1.$ (50)

Let $y(x) = y_0(x) + \varepsilon y_1(x) + \varepsilon^2 y_2(x).$ (51)

Therefore, $y''(x) = y_0''(x) + \varepsilon y_1''(x) + \varepsilon^2 y_2''(x).$ (52)

Since, $y(0) = y_0(0) + \varepsilon y_1(0) + \varepsilon^2 y_2(0) + \cdots \dots \dots \dots \dots \dots \dots \dots \dots \dots \dots \dots \dots$

$\therefore 1 = y_0(0) + \varepsilon y_1(0) + \varepsilon^2 y_2(0) + \cdots \dots \dots \dots \dots \dots \dots \dots \dots \dots \dots \dots \dots$

$y_0(0) = 1 \; and \; y_n(0) = 0, n \geq 1$

Also, $y'(0) = y_0'(0) + \varepsilon y_1'(0) + \varepsilon^2 y_2'(0) + \cdots \dots \dots \dots \dots \dots \dots \dots \dots$

$\therefore 1 = y_0'(0) + \varepsilon y_1'(0) + \varepsilon^2 y_2'(0) + \cdots \dots \dots \dots \dots \dots \dots \dots \dots$

$y_0'(0) = 1 \; and \; y_n'(0) = 0, n \geq 1$

Putting (51) and (52) in (50) we get

$y_0'' + \varepsilon y_1'' + \varepsilon^2 y_2'' + \dots \dots \dots = \varepsilon x[y_0 + \varepsilon y_1 + \varepsilon^2 y_2 + \cdots].$ (53)

Eq. (53) is an identity. It is true only when the coefficients of the like powers of ε from both sides are equal.

$y_0'' = 0, \quad y_0(0) = 1, y_0'(0) = 1$

$y_1'' = xy_0, \; y_1(0) = 0, y_1'(0) = 0$

$y_2'' = xy_1, \; y_2(0) = 0, y_2'(0) = 0$

Proceeding in this way, we have

$y_n'' = xy_{n-1}, \; y_n(0) = 0, y_n'(0) = 0.$

Now,

$y_0'' = 0$

$y_0' = c_1$

$\therefore y_0 = c_1 x + c_2$

$y_0(0) = c_1.0 + c_2$

$\therefore c_2 = 1$

$y_0' = c_1$

$y_0'(0) = c_1$

$\therefore c_1 = 1$

$\therefore y_0 = x + 1$

Again, $y_1'' = x$

$\therefore y_1 = \frac{x^4}{12} + \frac{x^3}{6} + c_1 x + c_2$

$y_1(0) = c_2$
$\therefore c_2 = 0$
And
$y_1'(0) = c_1$
$\therefore c_1 = 0$
$\therefore y_1 = \frac{x^4}{12} + \frac{x^3}{6}$
Again, $y_2'' = xy_1$
$y_2'' = \frac{x^5}{12} + \frac{x^4}{6}$
$\therefore y_2' = \frac{x^6}{72} + \frac{x^5}{30} + c_1$
$\therefore y_2 = \frac{x^7}{504} + \frac{x^6}{180} + c_1 x + c_2$
$y_2(0) = c_2$
$\therefore c_2 = 0$
And,
$y_2'(0) = c_1$
$\therefore c_1 = 0$
$\therefore y_2 = \frac{x^7}{504} + \frac{x^6}{180}$
$\therefore$ The required perturbation solution is
$y(x) = 1 + x + \varepsilon\left(\frac{x^3}{6} + \frac{x^4}{12}\right) + \varepsilon^2\left(\frac{x^6}{180} + \frac{x^7}{504}\right) + \cdots$

Example 2

$y'' = -e^{-x}y, \qquad y(0) = y'(0) = 1$

We want to find out its solution by different methods and show a comparison among those solutions.
Solution:

(i) The perfect solution given,

$$y'' = -e^{-x}y, \qquad y(0) = y'(0) = 1. \tag{54}$$

Using the substitution,

$$z = 2e^{-\frac{x}{2}}. \tag{55}$$

In the equation, we take the equation of the form,

$$z\frac{d^2y}{dz^2} + \frac{dy}{dz} + zy = 0. \tag{56}$$

Two linearly independent solutions of Eq. (56) are $J_0(z)$, and $Y_0(z)$. Therefore, the optimum output is given by,

$$y(x) = c_1 J_0(z) + c_2 Y_0(z), \qquad \text{Where, } z = 2e^{-\frac{x}{2}}. \tag{57}$$

This gives,

$$y'(x) = \{c_1 J'(z)\}\left(-e^{-\frac{x}{2}}\right). \tag{58}$$

For c_1 and c_2, in (57) and (58) finally we get,

$$c_1 = \frac{Y'_0(2) + Y_0(2)}{J_0(2)Y'_0(2) - J'_0(2)Y_0(2)}. \tag{59}$$

$$c_2 = -\frac{J'_0(2) + J_0(2)}{J_0(2)Y'_0(2) - J'_0(2)Y_0(2)}. \tag{60}$$

Putting these values in Eq. (57), we obtain the exact solution

$$y(x) = \frac{\{Y'_0(2) + Y_0(2)\}J_0\left(2e^{-\frac{x}{2}}\right) - \{J'_0(2) + J_0(2)\}Y_0\left(2e^{-\frac{x}{2}}\right)}{J_0(2)Y'_0(2) - J'_0(2)Y_0(2)}. \tag{61}$$

(ii) The perturbation Solution:

We change the problem into a perturbation problem by introducing the parameter ε in such a way that the unperturbed problem is solvable,

$Y" = -\varepsilon e^{-x} y \quad , \qquad y(0) = y'(0) = 1 \,.$ (62)

We take a perturbation expansion for y(x) of the form,

$$y(x) = \sum_{n=0}^{\infty} \varepsilon^n y_n(x) . \qquad (63)$$

Hence

$$y"(x) = \sum_{n=0}^{\infty} \varepsilon^n y"_n(x) . \qquad (64)$$

Now with the use of (63) and (64), (62) takes the form

$$\sum_{n=0}^{\infty} \varepsilon^n y_n''(x) = -\varepsilon e^{-x} \sum_{n=0}^{\infty} \varepsilon^n y_n(x)$$

i.e. $y_0'' + \varepsilon y_1'' + \varepsilon^2 y_2'' + \varepsilon^3 y_3'' + \cdots + \varepsilon^n y_n'' \ldots = -\varepsilon e^{-x} y_0 - \varepsilon^2 e^{-x} y_1 - \cdots - \varepsilon^n e^{-x} y_{n-1} - \cdots$ (65)

This gives a sequence,

$y_0'' = 0, y_o(0) = 1, y_0'(0) = 1.$ (66)

$y_n'' = e^{-x} y_{n-1}, \quad y_n(0) = 0, y_n'(0) = 0, \quad n \geq 1.$ (67)

Solving (67) we get, $y_0 = 1 + x$.

Now for $n = 1, \quad y_1'' = e^{-x} y_0 \,, \quad y_1(0) = 0, \quad y_1'(0) = 0$, whose solution is,

$y_1 = \int_0^x dt \int_0^t (1+s)(-e^{-s}) ds = 3 - 2x - (3+x)e^{-x}$.

For $n = 2\,, \; y_2'' = e^{-x} y_1\,, \; y_2(0) = 0, \; y_2'(0) = 0$, whose solution is,

$y_2 = \int_0^x dt \int_0^t (-e^{-u})[y_1(u)] du = -2 + \frac{3}{4}x + (1+2x)e^{-x} + (1+\frac{x}{4})e^{-2x}$.

We find,

$y_3 = \frac{41}{108} - \frac{1}{9}x + \left(\frac{1}{2} - \frac{3x}{4}\right) e^{-x} - \left(\frac{3}{4} + \frac{x}{2}\right) e^{-2x} - (\frac{7}{54} + \frac{x}{36}) e^{-3x}$.

and

$y(x) = y_0 + \varepsilon y_1 + \varepsilon^2 y_2 + \varepsilon^3 y_3$.

When $\varepsilon = 1$, we get the approximate solution up to fourth-term as,

$$y(x) = 1 + x + 3 - 2x - (3+x)e^{-x} - 2 + \frac{3}{4}x + (1+2x)e^{-x} + \left(1 + \frac{x}{4}\right) e^{-2x} + \frac{41}{108} - \frac{1}{9}x + \left(\frac{1}{2} - \frac{3x}{4}\right) e^{-x} - \left(\frac{3}{4} + \frac{x}{2}\right) e^{-2x} - \left(\frac{7}{54} + \frac{x}{36}\right) e^{-3x} \qquad (68)$$

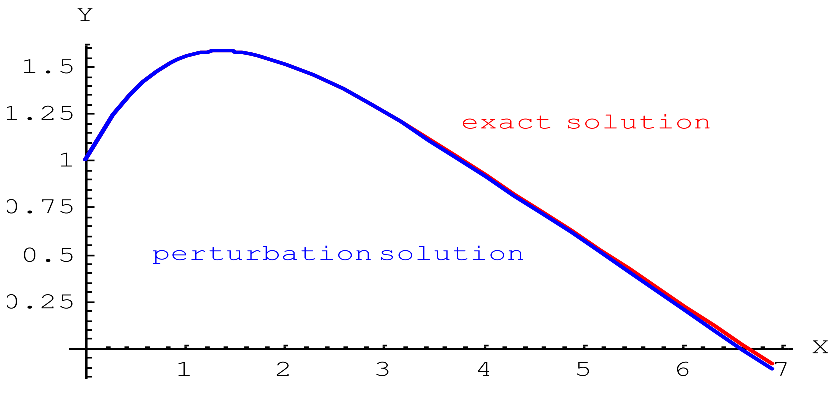

Fig. 4: A comparison of perturbation series approximations.

Example 3

An approximation solution of the formidable-looking non-linear two-point boundary value problem -

$$y''+y=\frac{Cosx}{3+y^2}\ :\ y(0)=y(\pi/2)=2\cdot \tag{69}$$

May be readily obtained using perturbation theory.

Since it is not possible to find the exact solution to this problem, it can also be solved numerically using nonlinear shooting with Newton's Method.

Perturbation solution

First, we introduce ε in such that the under-handed problem is solvable:

$$y''+y=\frac{\varepsilon Cosx}{3+y^2}\ :\ y(0)=y(\pi/2)=2\cdot$$

Or, $(x+y^2)(y''+y)=\varepsilon cosx.$ (A)

Secondly, let us assume that a perturbation expansion for $y(x)$ of the form

$$y(x)=\sum_{n=0}^{\infty}\varepsilon^n y_n(x)$$

$$y''(x)=\sum_{n=0}^{\infty}\varepsilon^n y''_n(x) \Rightarrow (3+y.y)\ (y''+y)=\varepsilon Cosx$$

$$\Rightarrow \left\{3+\sum_{n=0}^{\infty}\varepsilon^n y_n(x).\sum_{n=0}^{\infty}\varepsilon^n y_n(x)\right\}.\left\{\sum_{n=0}^{\infty}\varepsilon^n y''_n(x)+\sum_{n=0}^{\infty}\varepsilon^n y_n(x)\right\}=\ \varepsilon cosx$$

$$\Rightarrow \left(3+y_0.y_0+y_0.\varepsilon y_1 y_0\varepsilon^2 y_3+y_0\varepsilon^4 y_4+................\right)$$

$$\text{Again, }\left\{\sum_{n=0}^{\infty}\varepsilon^n y''_n(x)+\sum_{n=0}^{\infty}\varepsilon^n y_n(x)\right\}=\varepsilon Cosx$$

$$\Rightarrow \left(3+y^2{}_0+2\varepsilon y_0 y_1+2\varepsilon^3 y_0 y_3+2\varepsilon^4 y_0 y_4+..................\right)$$

$$\left\{\left(y''_0+y_0\right)+\varepsilon(y''_1+y_1)+\varepsilon^2(y''_2+y_2)+.........\right\}=\varepsilon Cosx\,.$$

Equating the coefficient of various powers of ε, we get.

$$y_0''+y_0=0, y_0(0)=2, y_0\left(\frac{\pi}{2}\right)=2. \tag{70}$$

$$\therefore\quad (3+y_0^2)(y_1''+y_1)=Cos\,x$$

$$y_1''+y_1=\frac{cosx}{3+y_0^2}, y_1(0)=0, y_1\left(\frac{\pi}{2}\right)=0. \tag{71}$$

The A.E of (70) is, $m^2+1=0 \Rightarrow$ m = ± i

$$\therefore\ y_c=c_1Cosx+c_2Sinx. \tag{72}$$

$$y_0\ (0)=c_1.1+c_2.0=2 \Rightarrow c_1=2$$

and $y_0(\pi/2)=c_1.1+c_2.1=2 \Rightarrow c_2=2$

(72) $\Rightarrow$ $y_0=2Cosx+2Sinx$.

(71) $\Rightarrow$ $y_1''+y_1=\dfrac{Cosx}{3+4\left(Cos^2x+Sin^2x+2SinxCosx\right)}$

$$y_1''+y_1=\frac{cosx}{7+4sin2x}. \tag{73}$$

$$\therefore\quad y_c=c_1Sinx+c_2Cosx$$

We assume that, $y_p(x)=v_1(x)sinx+v_2(x)cosx.$ (74)

$$\Rightarrow\ y_p'(x)=v_1(n)Cosx-v_2Sinx+v'_1(x)Sinx+v_2(x)Cosx$$

We impose the condition,

$$v_1(x)sinx+v_2(x)cosx=0. \tag{75}$$

$$\therefore\ y_p'(x)=v_1(x)Cosx-v_2(x)Sinx$$

$$y_p''(x)=-v_1(x)sinx-v_2(x)cosx+v_1(x)cosx-v_2(x)sinx. \tag{76}$$

Putting these values from (76) and (74) in (7) $\Rightarrow$

$$-v_1(x)Sinx - v_2(x)Cosx + v_1'(x)Cosx - v_2'(x)Sinx + v_1(x)Sinx + v_2(x)Sinx = \frac{Cosx}{7+4Sin2x}$$

$$\Rightarrow v_1(x)cosx - v_2(x)sinx = \frac{cosx}{7+4sin2x}. \qquad (77)$$

Now, (75) and (77) $\Rightarrow$

$$v_1'Sinx + v_2'(x)Cosx = 0 \text{ and } v_1'Sinx + v_2'Sinx = \frac{Cosx}{7+4Sin2x}.$$

$$\therefore \quad v_1'(x) = \frac{\frac{1}{2}(1+Cosx)}{7+4Sin2x}$$

$$\therefore \quad v_1'(x) = \frac{1}{2\sqrt{33}}\tan^{-1}\left(\frac{4+7\tan x}{\sqrt{33}}\right) + \frac{1}{16}\ln\left(7+4Sin2x\right)$$

$$\therefore \quad v'_2(x) = \frac{-SinxCosx}{7+4Sin2x} \Rightarrow v_2(x) = -\frac{x}{8} + \frac{7}{8\sqrt{33}}\tan^{-1}\left(\frac{7\tan x+4}{\sqrt{33}}\right).$$

The solution of Eq. (73) can be given in the following form:

$$y_1(x) = c_1Cosx + c_2Sinx - \frac{xCosx}{8} + \frac{7Cosx}{8\sqrt{33}}\tan^{-1}\left(\frac{4+7\tan x}{\sqrt{33}}\right)$$

$$+\frac{Sinx}{2\sqrt{33}}\tan^{-1}\left(\frac{4+7\tan x}{\sqrt{33}}\right) + \frac{Sinx}{16}\ln\left(7+4Sin2x\right)..................(C)$$

For, $y_1(0)=0, \quad y_1\left(\pi/2\right)=0$ then, $c_1 = -\frac{7}{8\sqrt{33}}\tan^{-1}\left(\frac{4}{\sqrt{33}}\right)$ and $c_2 = -\frac{\pi}{8\sqrt{33}} - \frac{1}{16}\ln(7)$.

A comparison of perturbation series approximation and Numerical solution of the boundary value problem in (A) is presented in the following figure. The graphs are one-term perturbation series approximation and two-term perturbation series approximation (dashed line) of the form in (C).

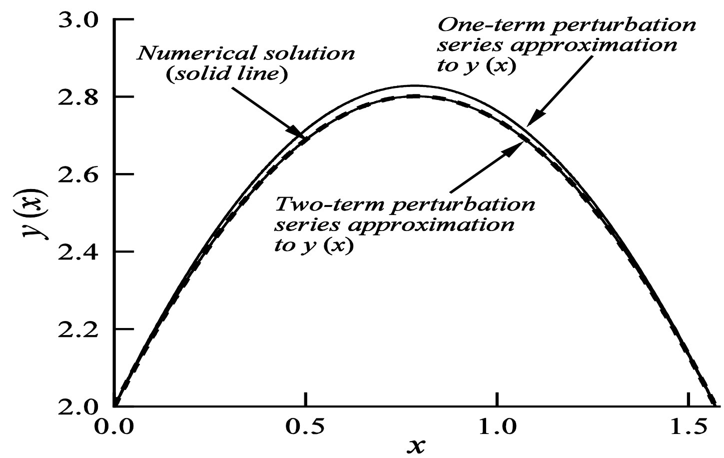

Fig. 4: A comparison of perturbation series approximation and numerical solution of the boundary-value problem in error.

CONCLUSION:

The present paper deals with the perturbation method and implies how much effective it is to solve a nonlinear differential equation involving initial as well as boundary conditions. Numerical solutions have been used to fulfill the investigation. Comparisons between numerical and perturbation solutions have been performed using Mathematica and Fortran Programming. Perturbation methods offer a valuable approach for approximating solutions to differential equations encountered in diverse scientific and engineering contexts. By systematically treating small deviations from known solutions, perturbation methods provide insights into system behavior, facilitate analytical predictions, and aid in the design of engineering systems. Through theoretical developments, numerical techniques, and practical applications, perturbation methods continue to play a vital role in advancing our understanding of complex dynamical systems.

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CONFLICTS OF INTEREST:

The authors declare that there are no conflicts of interest.

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